[This question paper contains 7 printed pages.]

2131 *Your Roll No.* ....

B.Sc. (Hons.) / I **C**

MATHEMATICS – Paper II

(Analysis - I)

(Admissions of 2009 and 2010)

Time: 3 Hours *Maximum Marks: 75*

(Write your Roll No. on the top immediately on receipt of this question paper.)

All *the questions are compulsory.*

1. Attempt any **two** parts :

(a) (i) If $x, a \in R$ and $\alpha > 0$ then show that $|x - a| < \alpha$ if and only if $a - \alpha < x < a + \alpha$.

(ii) Show that a number u is the infimum of a nonempty subset S of R if and only if u satisfies the conditions

(1) $s \geq u$ for all $s \in S$

(2) if $v > u$, then there exists $s' \in S$ such that $v > s'$ (2.5,3)

P.T.O.

(b) (i) If x and y are any real numbers with $x < y$, then show that there exists a rational number $r \in Q$ such that $x < r < y$.

(ii) Show that $\sup\left\{1 - \frac{1}{n} : n \in N\right\} = 1$ (3.2.5)

(c) (i) If $I_n = [a_n, b_n]$, $n \in N$ is a nested sequence of closed and bounded intervals in R, then show that there exists a $\xi \in R$ such that $\xi \in I_n \ \forall \ n \in N$.

(ii) Define limit point of a set in R. Determine the set of all limit points of the interval (0,1).

(3.2.5)

2. Attempt any **three** parts :

(a) Let (x_n) be a sequence of positive real numbers such that $\lim(x_{n+1}/x_n) = L < 1$. Show that there exists a number r with $0 < r < 1$ such that $0 < x_n < r^n$ for all sufficiently large $n \in N$. Hence, prove that

$\lim(x_n) = 0$ (5)

(b) Determine the following limits and also state all theorems used to evaluate these limits :

(i) $\lim_{n\to\infty} \frac{x^n}{n!}$ for any real number x

(ii) $\lim_{n\to\infty} (n!)^{1/n^2}$ (3.2)

(c) (i) If (x_n) is a sequence of real numbers, show that there is a subsequence of (x_n) that is monotone

(ii) Prove that $\lim_{n\to\infty} (b^{1/n}) = 1$ for $b > 1$. (3.2)

(d) (i) Prove that every Cauchy sequence in R is convergent.

(ii) Using the definition, show that the sequence $((-1)^n)$ is not a Cauchy sequence. (2.3)

3. Attempt any **two** parts :

(a) Suppose that $\sum a_k$ and $\sum b_k$ are two infinite series of positive real numbers such that

$$\lim_{k\to\infty} \frac{a_k}{b_k} \neq 0$$

Then show that $\sum a_k$ is convergent if and only if

$\sum b_k$ is convergent. Hence or otherwise prove

that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ is divergent. (4.2)

(b) Examine the convergence and absolute convergence of the following series

(i) $\sum_{n=1}^{\infty} \frac{1}{3^n + x^n}$ for all positive values of x.

(ii) $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{\sqrt{n^5}} + \frac{1}{\sqrt{(n+1)^5}} \right)$ (3.3)

(c) Give the statement of ratio test for an infinite series $\sum a_n$. Hence or otherwise examine the convergence of the series

$\sum 2^{[illegible]}$ and $\sum_{n=1}^{\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ (2.2.2)

4. Attempt any **three** parts :

(a) (i) Determine a condition on $|x-4|$ that will assume that $|\sqrt{x}-2| < 10^{-2}$.

(ii) Use the definition of limit to show that $\lim_{x \to 2}(x^2 + 3x) = 10$. (3.2)

(b) (i) Let $c \in R$ and let $f : R \to R$ be such that $\lim_{x \to c}(f(x))^2 = L$. Show that if $L = 0$ then $\lim_{x \to c} f(x) = 0$.

(ii) Show that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

(3.2)

(c) (i) Let $f : A \to R$ be a function. If (x_n) is a sequence in A that converges to c such that the sequence $(f(x_n))$ converges to f(C). Then show that f is continuous at point c.

(ii) Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is continuous only at $x = ½$.

(3.2)

(d) (i) Let $I = [a,b]$ be a closed bounded interval and let $f : I \to R$ be continuous on I. Prove that f has an absolute maximum on I.

(ii) Give an example of a function $f : [0,1] \to R$ that is discontinuous at every point of [0,1] but |f| is continuous on [0,1]. (3.2)

5. Attempt any two parts:

(a) Show that the function $f(x) = 1/x^2$ is uniformly continuous on $A = [1, \infty)$ but is not uniformly continuous on $B = (0, \infty)$. (3,3)

(b) Let $f : I \to R$ be differentiable on an interval I. Prove that if $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I. Is the converse true? Justify your answer. (4,2)

(b) (i) Use Mean Value theorem to prove:

$$|\sin x - \sin y| \leq |x - y|, \text{ for all } x, y \in R$$

(ii) Let f and g be differentiable function on (a,b) such that $f' = g'$ on (a,b). Then there exists a constant c such that

$$f(x) = g(x) + c \text{ for all } x \in (a,b)$$ (3,3)

6. Attempt any two parts:

(a) (i) Obtain Maclaurin's series expansion of

$$f(x) = \cos x, \quad x \in R$$

(ii) Show that $0 < [\log(1+x)]^{-1} - x^{-1} < 1$ whenever $x > 0$. (2,3)

(b) Show that if $x > 0$ then

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 \le \sqrt{1+x} \le 1 + x/2$$

and use this inequality to approximate $\sqrt{2}$. (5)

(c) (i) Approximate the number e with error less than 10

(ii) Let I be an open interval and let $f : I \to R$ have a second derivative on I. Prove that if f is a convex function on I then $f''(x) \ge 0$ for all $x \in I$. (2,3)