[This question paper contains 4 printed pages.]

2132 *Your Roll No.* ....

B.Sc. (Hons.) / I **C**

MATHEMATICS – Paper III

(Algebra – I)

(Admissions of 2009 and 2010)

Time: 3 Hours *Maximum Marks: 75*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Do any **two** *parts from each question.*

(a) The diagonals of the square intersect at A and B is one vertex of the square. Prove that the four vertices of the square are represented by the numbers :

$a + (b - a),\ a + i(b - a),\ a - (b - a),\ a - i(b - a)$,

where the points A, B are represented by the complex numbers a, b. (6)

(b) Solve the equation

$x^4 + 15x^3 + 70x^2 + 120x + 64 = 0$

the roots being in G.P. (6)

(c) Using Descartes' rule of signs, show that the equation

$x^6 - 2x^5 + x^4 + x^2 - 2x + 1 = 0$

must have at least two complex roots. (6)

P.T.O.

2. (a) Let $S = \{1, 2, 3\}$. Determine which of the following binary relations are equivalence relation on S

(i) $R = \{(1, 1), (1, 2), (3, 2), (3, 3), (2, 3), (2, 1)\}$

(ii) $R = \{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (2, 3), (3, 1), (1, 3)\}$ (5)

(b) Let $A = \{x \in R / x \neq 4\}$, $f(x) = 1 + \frac{1}{x - 4}$.

(i) Show that $f : A \to R$ is one-to-one.

(ii) Find the range of f and its inverse. (5)

(c) Define cardinality of a set. Show that N and $N \cup \{0\}$ have same cardinality. (5)

3. (a) Let $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in Z \right\}$ under addition.

Let $H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G \middle| a + b + c + d = 0 \right\}$. Prove that H is a subgroup of G. (7½)

(b) Let $G = \langle a \rangle$ be a cyclic group of order n. Prove that $G = \langle a^k \rangle$ if and only if $\gcd(n, k) = 1$. Find all the generators of Z_{30}. (7½)

(c) Prove that

(i) the center of a group G is an Abelian subgroup of G.

(ii) G is Abelian if and only if $G = Z(G)$. (7½)

4. (a) Give an example of each of the following with justification :

   (i) Ring which is not commutative but has a subring which is commutative.

   (ii) Ring which has no unity but has a subring which has unity. (4)

   (b) Show that a finite integral domain is a field. (4)

   (c) Prove that a nonempty subset of a ring is a subring if it is closed under subtraction and multiplication. (4)

5. (a) Find the solution set of the given nonhomogenous system in parametric vector form

$$3x_1 + 5x_2 - 4x_3 = 7$$
$$-3x_1 - 2x_2 + 4x_3 = -1$$
$$6x_1 + x_2 - 8x_3 = -4$$

(7½)

   (b) (i) Show that the transformation $T : R^2 \to R^2$ defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear.

   (ii) Let T be a linear transformation whose standard matrix is

$$\begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

   Does T map R^4 onto R^3? Is T a one-to-one mapping? (7½)

(c) (i) Determine if the columns of the matrix form a linearly independent set. Justify.

$$\begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 4 \end{pmatrix}$$

(ii) If v_1, v_2, v_3, v_4 are in R^4 and $\{v_1, v_2, v_3\}$ is linearly dependent, then show that $\{v_1, v_2, v_3, v_4\}$ is linearly dependent. (7½)

6. (a) Find the bases for the row space, the column space and the null space of the matrix

$$\begin{pmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{pmatrix}$$ (7½)

(b) Let

$$v_1 = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix},\ v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},\ x = \begin{pmatrix} 3 \\ 12 \\ 7 \end{pmatrix}.$$

(i) Show that $B = \{v_1, v_2\}$ is a basis for $H = \text{span}\{v_1, v_2\}$.

(ii) Determine if x is in H and, if it is, find the coordinate vector of x relative to B. (7½)

(c) Find the characteristic equation of

$$\begin{pmatrix} 5 & 2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Also find the eigen values. (7½)