[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 1875 **C** **Roll No............**

Unique Paper Code : 235201

Name of the Course : B.Sc. (Hons.) MATHEMATICS

Name of the Paper : Differential Equations and Mathematical Modelling – I (MAHT – 201)

Semester : II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions.
3. Use of scientific calculator is allowed.

SECTION I

1. Attempt any **three** of the following : (5+5+5)

(a) Find explicit particular solution of the initial value problem

$$(\tan x)\frac{dy}{dx} = y, \quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

(b) Solve the differential equation

$$x^2\frac{dy}{dx} + 2xy = 5y^3.$$

(c) Solve the differential equation

$$\frac{dy}{dx} = (x + y + 3)^2.$$

(d) Solve the differential equation

$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}.$$

P.T.O.

(e) Solve the differential equation

$$(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0.$$

by finding the integrating factor of the form $x^p y^q$.

2. Attempt any **two** of the following : (5+5)

(a) Carbon taken from a purported relic of the time of Christ contained 4.6×10^{10} atoms of ^{14}C per gram. Carbon extracted from a present-day specimen of the same substance contained 5×10^{10} atoms of ^{14}C per gram. Compute the approximate age of the relic. (Decay constant k = 0.0001216 if time is measured in years)

(b) A water tank has the shape obtained by revolving the curve $y = x^{4/3}$ around the y-axis. A plug at the bottom is removed at 12 noon, when the depth of water in the tank is 12ft. At 1 pm the depth of water is 6 ft. When will the tank be empty ?

(c) A motorboat starts from rest (initial velocity $v(0) = v_0 = 0$). Its motor provides a constant acceleration of $4 ft/s^2$, but water resistance causes a deceleration of $v^2/400\ ft/s^2$. Find v when t = 10s, and also find the *limiting velocity* as $t \to +\infty$ (that is, the maximum possible speed of the boat).

SECTION II

3. Attempt any **two** of the following : (8+8)

(a) A public bar opens at 6 pm and is rapidly filled with clients of whom the majority are smokers. The bar is equipped with ventilators which exchange the smoke-air mixture with fresh air. Cigarette smoke contains 4% carbon monoxide and a prolonged exposure to a concentration of more than 0.012% can be fatal. The bar has a floor area of 20 m by 15 m, and a height of 4 m. It is estimated that smoke enters the room at a constant rate of $0.006\ m^3/min$, and that the ventilators remove the mixture of smoke and air at 10 times the rate at which smoke is produced. The problem is to establish a wise time to leave the bar. That is, sometime before the concentration of carbon monoxide reaches the lethal limit.

(i) Starting from the word equation or a compartmental diagram formulate the differential equation for the changing concentration of carbon monoxide in the bar over time.

(ii) By solving the equation above, establish at what time the lethal limit will be reached.

(b) Consider the logistic differential equation

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right),$$

where X(t) is the measure of the population size, r is the growth rate and K is the carrying capacity.

(i) Solve the logistic differential equation to obtain the solution

$$X = \frac{X_0}{(K - X_0)} e^{rt}(K - X), \text{ where } X(0) = X_0$$

(ii) Show that $X \to K$ as $t \to \infty$.

(c) In a fish farm, fish are harvested at a constant rate of 2100 fish per week. The per-capita death rate for the fish is 0.2 fish per day per fish, and the per-capita birth rate is 0.7 fish per day per fish.

(i) Write down a word equation describing the rate of change of the fish population. Hence obtain a differential equation for the number of fish. (Define all the symbols introduced.)

(ii) If the fish population at a given time is 240000, give an estimate of the total number of fish after one week.

(iii) Determine if there are any values for which the fish population is in equilibrium.

SECTION III

4. Attempt any **three** of the following : (6+6+6)

(a) Consider the differential equation :

$$x^2y'' + 2xy' - 12y = 0.$$

Substitute $v = \ln x$ ($x>0$) in the above equation and solve.

(b) Show first that the three solutions

$$y_1(x) = e^x, \; y_2(x) = e^{2x}, \quad \text{and} \quad y_3(x) = e^{3x}$$

of the third order differential equation

$$y^{(3)} - 6y'' + 11y' - 6y = 0$$

are linearly independent on the open interval. Then find a particular solution of the above differential equation that satisfies the initial conditions

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 3.$$

(c) Use the method of undetermined coefficients to find the particular solution of the differential equation :

$$y'' + y' + y = \sin^2 x.$$

(d) Use the method of variation of parameters to find a particular solution of the differential equation

$$y'' - 2y' - 8y = 3e^{-2x}.$$

(e) A body with mass $m = \frac{1}{2}$ kg is attached to the end of a spring that is stretched 2 m by a force of 100 N. It is set in motion with initial position $x_0 = 1$ m and an initial velocity $v_0 = -5$ m/s (Note that these initial conditions indicate that the body is displaced to the right and is moving to the left at time $t = 0$). Find the position function of the body and the amplitude of the motion of the body.

SECTION IV

5. Attempt any **one** of the following : (16)

(a) A simple model describing a jungle warfare, with one army exposed to random fire and the other to aimed fire, is given by the coupled differential equations

$$\frac{dR}{dt} = -c_1RB, \quad \frac{dB}{dt} = -a_2R,$$

where c_1 and a_2 are positive constants.

(i) Use the chain rule to find a relation between B and R, given the initial numbers of blue and red soldiers are b_0 and r_0 respectively.

(ii) Hence sketch some typical phase-plane trajectories. Give directions of travel along the trajectories.

(iii) Given that initially, both the red and blue armies have 1000 soldiers, and the constants c_1 and a_2 are 10^{-4} and 10^{-1}, respectively, determine how many soldiers are left if the battle is fought so that all the soldiers of one army are killed.

(iv) In this model, one of the armies is hidden whereas the other is visible to their enemy. Which is the hidden army ? Give reasons for your answer.

(b) Consider the Lotka-Volterra model describing the simple predator-prey model

$$\frac{dX}{dt} = \beta_1X - c_1XY, \quad \frac{dY}{dt} = c_2XY - \alpha_2Y,$$

(i) Solve the model in the limiting case of prey with no predator, and of predators with no prey.

(ii) Find all equilibrium points.

(iii) Find directions of trajectories in the phase plane.

(iv) Modify the model by including density dependent growth for the prey with carrying capacity K.