[This question paper contains 4 printed pages.]

2105

Your Roll No. ....

B.Sc. (Hons.) / II **C**

MATHEMATICS – Paper V

(Analysis – 2)

(Admissions of 2009 and 2010)

Time: 3 Hours *Maximum Marks: 75*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt **All** *questions. Attempt two parts from each question. Internal choice is given. Marks for each question and distribution of marks for different parts of every equation are indicated.*

1. (a) If f and g are integrable on [a, b] then show that f+g is integrable on [a, b] and

$$\int_a^b f + g = \int_a^b f + \int_a^b g \qquad (6)$$

(b) State and Prove Fundamental Theorem of Calculus I. (6)

(c) Show that a bounded function f on the closed interval [a, b] is integrable if and only if for each $\epsilon > 0$, there exists a partition P of [a, b] such that $U(f, P) - L(f, P) < \epsilon$. (6)

2. (a) (i) Show that if a > 0 then the convergence of the sequence $\left\langle \frac{nx}{1+nx} \right\rangle$, $x \geq 0$ is uniform on the interval [a, ∞[but is not uniform on [0, ∞[. (6)

P.T.O.

(ii) Show that the Exponential function E is strictly increasing on $\mathbb{R}$. (3)

(b) (i) Let $\langle f_n \rangle$ be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose $\langle f_n \rangle \to f$ uniformly on A. Prove that f is continuous on A. (6)

(ii) Show that the sequence $\left\langle \frac{x^n}{1+x^n} \right\rangle$ does not converge uniformly on $[0,2]$ by showing that the limit function is not continuous on $[0, 2]$. (3)

(c) (i) Show that there exists a root ϑ of the cosine function C in the interval $\left(\sqrt{2}, \sqrt{3}\right)$ such that $C(x) > 0$ for all $x \in [0, \vartheta[$ and 2ϑ is the smallest positive root of the sine function S. (6)

(ii) Show that if $a > 0$, the series $\sum_{n=1}^{\infty} (nx)^{-2}$, $x \neq 0$ is uniformly convergent for $|x| \geq a$. (3)

3. (a) Show that :

$$\int_1^{\infty} \frac{\sin t}{t}\, dt$$

is not absolutely Convergent. (5)

(b) Examine the Convergence of the improper integrals :

(i) $\int_1^x \frac{1}{\log t}\,dt$

(ii) $\int_1^x \frac{2^t}{t^2}\,dt$ (5)

(c) Find the radius of Convergence of the Power series :

(i) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$

(ii) $\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$ (5)

4. (a) Write $\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\,dy\,dx$ as an integral over a region. Sketch the region. Reverse the order of integration and then evaluate. (6)

(b) By using the change of variables $u = x - y$, $v = x + y$, show that

$$\iint_D \cos\left(\frac{x-y}{x+y}\right) dx\,dy = \frac{\sin 1}{2}$$

where D is the region bounded by $x + y = 1$, $x = 0$, $y = 0$. (6)

(c) Evaluate

$$\iiint_W \frac{dx\,dy\,dz}{\left(x^2+y^2+z^2\right)^{3/2}}$$

where W is the solid bounded by the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$ and $a > b > 0$. (6)

P.T.O.

5. (a) Let C be the perimeter of the Unit square $[0,1] \times [0,1]$ in the plane, traversed in the counter clockwise direction. Evaluate the line integral

$$\int_C \vec{F} \cdot \vec{ds}$$

where $\vec{F}(x,y) = x^2\hat{i} + xy\hat{j}$ (6)

(b) Find the area of the graph of the function

$$f(x,y) = \frac{2}{3}\left(x^{3/2} + y^{3/2}\right)$$

over the domain $[0,1] \times [0,1]$. (6)

(c) Find the flux of $\vec{F}(x,y,z) = 3xy^2\hat{i} + 3x^2y\hat{j} + z^3\hat{k}$ out of the unit sphere. (6)

6. (a) State Green's theorem. Use it to find the work done by the force field $(3x + 4y)\hat{i} + (8x + 9y)\hat{j}$ on a particle that moves once around the ellipse $4x^2 + 9y^2 = 36$. (5½)

(b) State Stokes's theorem. Use it to evaluate :

$$\int_C \vec{F} \cdot \vec{ds}$$

where $\vec{F}(x,y,z) = 2x\hat{i} - y\hat{j} + (x - z)\hat{k}$ and C consists of straight lines joining (1, 0, 1), (0, 1, 0) and (0, 0, 1). (5½)

(c) State and prove Gauss Divergence Theorem. (5½)

(3,000)