[This question paper contains 8 printed pages.]

2133 *Your Roll No.* ....

B.Sc. (Hons.) / II **C**

MATHEMATICS – Paper IV

(Differential Equations and Mathematical Modelling – I)

(Admissions of 2009 and onwards)

Time : 3 Hours *Maximum Marks : 70*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt **All** *questions.*

Use of scientific calculator is allowed.

1. (a) Solve any one differential equation.

(i) $(x^2 - 3y^2)\,dx + 2xy\,dy = 0$

(ii) $x\dfrac{dy}{dx} + (2x + 1)y = xe^{-2x}$ (6)

(b) Suppose that a body moves through a resisting medium with resistance proportional to its velocity i.e. $\dfrac{dv}{dt} = -kv$. Show that its velocity and position at any time 't' are given by

P.T.O.

$$v(t) = v_0 e^{-kt} \text{ and}$$

$$x(t) = x_0 + \left(\frac{v_0}{k}\right)\left(1 - e^{-kt}\right) \qquad (6)$$

(c) Verify that the functions $y_1(x) = e^x$ and $y_2(x) = x\,e^x$ are the solutions of the differential equation $y'' - 2y' + y = 0$, and then find a solution satisfying the initial conditions $y(0) = 3$, $y'(0) = 1$. (6)

OR

(a) Solve any one differential equation.

(i) $(x + 2y + 3)\,dx + (2x + 4y - 1)\,dy = 0$

(ii) $\frac{dy}{dx} + y = xy^3$ (6)

(b) Solve the initial value problem :

$y'' + 2y' + y = 0$, $y(0) = 5$, $y'(0) = -3$ (6)

(c) A hemispherical bowl has top radius 4 ft and at time $t = 0$ is full of water. At that moment a circular hole with diameter 1 inch is opened in the bottom of the tank. How long will it take for all the water to drain from the tank ? (take $g = 32$ ft/sec^2). (6)

2. (a) Write down the word equations which describe the movement of the drug between the two compartments in the body, the G.I. tract and the bloodstream, when a patient takes a single pill.

   (i) From the word equation, develop a differential equation system which describe this process, defining all variables and parameters as required.

   (ii) The constants of proportionality associated with the rates at which the drug diffuses from G.I. tract into the bloodstream and then is removed from the bloodstream, are 0.72 $hour^{-1}$ and 0.15 $hour^{-1}$ respectively. Suppose initially the amount of drug in G.I. tract is 0.0001 mg and none in the bloodstream determine the level of drug in the bloodstream after 6 hours. (8)

(b) In the view of potentially disastrous effects of overfishing causing a population to become extinct, some governments impose quotas which vary depending on estimates of the population at the current time. One harvesting model that takes this into account is

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right) - h_0X$$

(i) Show that the only non-zero equilibrium population is

$$X_e = K\left(1 - \frac{h}{r}\right)$$

(ii) At what critical harvesting rate can be extinction occur ? (6)

OR

(a) Consider the American system of two lakes: Lake Erie feeding into Lake Ontario. Assuming that the volume in each lake to remain constant and that Lake Erie is the only source of pollution for Lake Ontario

(i) Write down a differential equation describing the concentration of pollution in each of the two lakes, using the variables V for volume, F for flow, c(t) for concentration at time t and subscripts 1 for Lake Erie and 2 for Lake Ontario.

(ii) Suppose that only unpolluted water flows into Lake Erie. How does this change the model proposed ?

(iii) Solve the system of equations to get expression for the pollution concentration $c_1(t)$ and $c_2(t)$. (8)

(b) Consider the population of a country. Assume constant per-capita birth and death rates and that the population follows an exponential growth (or decay) process. Assume there to be significant immigration and emigration of people into and out of the country. Give clear explanations of how the differential equations are obtained from the word equations.

(i) Assuming the overall immigration and emigration rates are constant, formulate a single differential equation to describe the population size over time.

(ii) Suppose instead that all immigration and emigration occurs with a neighbouring country, such that the net movement from one country to the other is proportional to the population difference between the two countries and such that people move to the country with the larger population. Formulate a coupled system of equations as a model for this situation. (6)

3. (a) Find the particular solution of the differential equation $y'' + 3y' + 4y = 3x + 2$. (7)

(b) Use the method of variation of parameters to find the solution of the equation

$$y'' + y = \tan x. \quad (7)$$

OR

(a) Use the method of variation of parameters to find the solution of the equation

$$y'' + y = \operatorname{cosec}^2 x \qquad (7)$$

(b) Use the method of undetermined coefficients to find the solution of the equation

$$y'' + 4y = 3x^3 \qquad (7)$$

4. The predator-prey equations with additional deaths by DDT are

$$\frac{dX}{dt} = \beta_1 X - c_1 XY - p_1 X \quad \frac{dY}{dt} = -\alpha_2 Y + c_2 XY - p_2 Y,$$

where all parameters are positive constants, $X(t)$ denotes the prey population and $Y(t)$ denotes the predator populations at any time t.

(i) Find all the equilibrium points.

(ii) What effect does the DDT have on the non-zero equilibrium populations compared with the case when there is no DDT ?

(iii) Show that the predator fraction of the total average prey predator population is given by

$$f = \frac{1}{1 + \left(\frac{c_1}{c_2} \frac{(\alpha_2 + p_2)}{(\beta_1 - p_1)} \right)} \qquad (10)$$

OR

Consider a disease where all those infected remain contagious for life. A model describing this is given by the differential equations.

$$\frac{dS}{dt} = -\beta SI, \ \frac{dI}{dt} = \beta SI$$

where β is a positive constant. S(t) denotes the number of susceptibles and I(t) denotes the number of infectives at time t.

(i) Use the chain rule to find a relation between S and I.

(ii) Obtain and sketch the phase-plane curves. Determine the direction of travel along the trajectories.

(iii) Using this model, is it possible for all the susceptible to be infected. (10)

5. (a) Find the inverse Laplace transform of

$$G(s) = \frac{1}{s^2(s-a)} \quad (6)$$

(b) Apply the method of Frobenius to the equation

$$xy'' + 2y' + xy = 0$$

to derive its general solution. (8)

OR

(a) Show that : (6)

$$L\left[t^n e^{at}\right] = \frac{n}{(s-a)} L\left[t^{n-1} e^{at}\right] = \frac{n!}{(s-a)^{n+1}}$$

for $n = 1, 2, \ldots..$

(b) Use power series to solve the initial value problem :

$$y'' + xy' - 2y = 0,$$

$$y(0) = 1, \; y'(0) = 0.$$ (8)