[This question paper contains 4 printed pages.]

2136 *Your Roll No.* ....

B.Sc. (Hons.) / II **C**

MATHEMATICS – Paper VII

(Algebra – II)

(Admissions of 2009 and onwards)

Time: 3 Hours *Maximum Marks: 75*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt **two** *parts from each question.*
All *questions are compulsory.*

1. (a) Every permutation of a finite set can be written as a cycle or as a product of disjoint cycles.

   (b) Suppose that $|G| = pq$, where p and q are prime. Prove that every proper subgroup of G is cyclic.

   (c) Let N be a normal subgroup of G and let H be a subgroup of G. If N is a normal subgroup of H, prove that H/N is a normal subgroup of G/N if and only if H is a normal subgroup of G.

(2×6=12)

P.T.O.

2. (a) Everey group is isomorphic to a group of permutations.

(b) Prove that there is no homomorphism from $Z_8 \oplus Z_2$ onto $Z_4 \oplus Z_4$.

(c) If G is non – abelian, show that Aut(G) is not cyclic. (2×6=12)

3. (a) Let D be an integral domain. Then there exists a field F (called the field of quotients of D) that contains a subring isomorphic to D.

(b) If n is a positive integer, show that $\langle n \rangle = nZ$ is a prime ideal of Z if and only if n is prime.

(c) Let ϕ be a homomorphism from a ring R into a ring S. Let A be an ideal of R and B be an ideal of S. then

(i) If ϕ is onto, then $\phi(A)$ is an ideal of S.

(ii) $\phi^{-1}(B) = \{r \in R : \phi(r) \in B\}$ is an ideal of R. (2×6=12)

4. (a) If D is an integral domain, then D[x] is an integral domain. Let F be a field, then F[x] is a principal ideal domain.

(b) Show that $x^2 + x + 4$ is a irreducible over Z_{11}.

(c) Let D be a principal ideal domain and let $p \in D$. Prove that $\langle p \rangle$ is a maximal ideal in D if and only if p is irreducible. (2×6½=13)

5. (a) Prove that if W_1 and W_2 are finite - dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite - dimensional and

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

(b) Let V and W be vector spaces over F, and suppose that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V. For $w_1, w_2, \ldots, w_n$ in W, there exists exactly one linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$.

(c) Let V and W be finite - dimensional vector spaces over F of dimensions n and m respectively, and let β and γ be ordered bases for V and W respectively. Then the function $\phi : L(V, W) \to M_{m \times n}(F)$, defined by $\phi(T) = [T]_\beta^\gamma$ for $T \in L(V, W)$, is an isomorphism, where $L(V, W)$ is the vector space of all linear transformations from V into W. (2×6½=13)

6. (a) Suppose that V is a finite - dimensional vector space with ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$. For each $i = 1, 2, \ldots, n$ define

$f_i(v) = a_i$, where $[v]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ is the coordinate vector of v relative to β. Prove that f_i is a linear functional on V. Let $\beta^* = \{f_1, f_2, \ldots, f_n\}$. Then β^* is an ordered basis for V^* and for any

$f \in V^*$, we have $f = \sum_{i=1}^{n} f(v_i) f_i$

(b) Let T be a linear operator on R^3 defined by

$$T\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4a_1 + a_3 \\ 2a_1 + 3a_2 + 2a_3 \\ a_1 + 4a_3 \end{bmatrix}$$

Determine the eigenspaces of T corresponding to each eigenvalue and hence find a basis for R^3 consisting of eigenvectors of T.

(c) Let T be a linear operator on a finite - dimensional vector space V and suppose that $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$, where W_i is T – invariant subspace of V for each $i (1 \le i \le k)$. Suppose that $f_i(t)$ is the characteristic polynomial of $T_{W_i} (1 \le i \le k)$, the restriction of T on W_i. Then $f_1(t) f_2(t) \ldots f_k(t)$ is the characteristic polynomial of T. (2×6½=13)