[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 1881 **C** **Roll No............**

Unique Paper Code : 235402

Name of the Course : B.Sc. (Hons.) Mathematics

Name of the Paper : IV.2 (ANALYSIS III) (MAHT–402)

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory.

1. (a) Show that a bounded function f on [a, b] is integrable if and only if for each $\varepsilon > 0$ there exists a partition P of [a, b] such that $U(f, P) - L(f, P) < \varepsilon$. (6)

   (b) Let f be a continuous function on $\mathbb{R}$ and define

   $$F(x) = \int_{x-1}^{x+1} f(t)dt \quad \text{for } x \in \mathbb{R}$$

   Show that F is differentiable on $\mathbb{R}$ and compute F'. (6)

   (c) State Fundamental Theorem of Calculus II. Find $\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}\,dt$. (6)

2. (a) Show that if f is integrable on [a, b] then |f| is integrable on [a, b] and

   $$\left|\int_a^b f\right| \le \int_a^b |f|. \quad (4,2)$$

   (b) Let f be defined on [0, b] as

   $$f(x) = \begin{cases} x, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

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Calculate upper and lower Darboux integrals for f on [0,b]. Is f integrable on [0, b] ? (6)

(c) Let f be a bounded function on [a, b]. If P and Q are partitions of [a, b] such that $P \subseteq Q$, where Q contains exactly one point extra than the points of P, then prove that

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P). \quad (6)$$

3. (a) Examine the convergence of the following improper integrals

(i) $\int_0^{\infty} \frac{dt}{t^p}$, $p \in \mathbb{R}$ (ii) $\int_{0+}^{1} \frac{dt}{\sqrt{t(t+1)}}$ (iii) $\int_1^{\infty} e^{t^2} dt$ (2,2,2)

(b) Show that the improper integral $\int_1^{\infty} \frac{\cos t}{t} dt$ is convergent but not absolutely convergent. (2,4)

(c) Show that the improper integral $\int_{0+}^{1} t^{p-1}(1-t)^{q-1} dt$ converges if and only if $p > 0,\ q > 0$. (6)

4. (a) Show that a sequence $\langle f_n \rangle$ of bounded functions on $A \subseteq \mathbb{R}$ converges uniformly on A to f if and only if $\|f_n - f\|_A \to 0$. (5)

(b) Let $f_n(x) = \frac{x}{1 + nx^2}$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$..

Show that $\langle f_n \rangle$ converges uniformly on $\mathbb{R}$. (5)

(c) Let $f_n(x) = \frac{nx}{1 + n^2x^2}$ for $x \geq 0$.

Show that the sequence $\langle f_n \rangle$ converges only pointwise on $[0, \infty)$ and converges uniformly on $[a, \infty)$, $a > 0$. (5)

5. (a) Let $f_n(x) = \dfrac{nx}{1 + nx}$ for $x \in [0,1]$, $n \in \mathbb{N}$.

Show that $\langle f_n \rangle$ converges non-uniformly to an integrable function f on [0,1].

Also examine the relationship between $\int_0^1 f(x)dx$ and $\lim_{n \to \infty} \int_0^1 f_n(x)dx$. (5½)

(b) Show that series of functions

$$\sum f_n(x) = \sum \frac{1}{x^n + 1}$$

converges only pointwise on $(1, \infty)$ and uniformly on $[a, \infty)$, $a > 1$. (5½)

(c) Let $\langle f_n \rangle$ be a sequence of real valued functions on $A \subseteq \mathbb{R}$. Show that $\sum f_n$ converges uniformly on A if and only if for each $\varepsilon > 0$, $\exists\, M(\varepsilon) \in \mathbb{N}$ such that

$$|f_{n+1}(x) + f_{n+2}(x) + \ldots + f_m(x)| < \varepsilon$$

for all $x \in A$ and for all $m > n \geq M(\varepsilon)$. (5½)

6. (a) (i) Show that power series $\sum_{n=0}^{\infty} a_n x^n$, with radius of convergence R, $0 < R \leq \infty$ converges uniformly to a continuous function on $[-R_1, R_1]$, $0 < R_1 < R$. (6)

(ii) Show that the function

$$L(y) = \int_1^y \frac{dt}{t},\ y \in (0, \infty)$$

is well defined and is differentiable on $(0, \infty)$ with

$$L'(y) = \frac{1}{y},\ y \in (0, \infty).$$ (3)

(b) (i) If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then the power series

$$\sum_{n=1}^{\infty} na_n x^{n-1} \text{ and } \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

also have radius of convergence R. (6)

(ii) Show that the function

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \ x \in \mathbb{R}$$

is strictly increasing on $\mathbb{R}$ and $\lim_{x\to\infty} E(x) = \infty$. (3)

(c) (i) Show that

$$\text{arc tan } x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \text{ for } |x| < 1 \quad (3)$$

(ii) If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $|x| < R$ then $a_k = \frac{f^k(0)}{k!}$ $\forall k \geq 0$. (3)

(iii) Show that the Logarithmic function

$L(y) = \int_1^y \frac{dt}{t}$, $y \in (0, \infty)$ satisfies the relation

$L(yz) = L(y) + L(z)$ $\forall$ $y, z \in (0, \infty)$. (3)