[This question paper contains 2 printed pages.]

Sr. No. of Question Paper : 1882 C Roll No............

Unique Paper Code : 235404

Name of the Course : B.Sc. (Hons.) Mathematics

Name of the Paper : MAHT 403 : Algebra III

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.

1. (a) Define a subring and an ideal of a ring. Prove that the intersection of two ideals of a ring is an ideal. What about their union ? Justify your answer.

   (b) Define a field and prove that a finite integral domain is a field. Can we drop the condition of 'finiteness' in the hypothesis ? Justify your answer.

   (c) Define the characteristic of a ring. Prove that the characteristic of an integral domain is either zero or a prime number. What is the characteristic of $\mathbf{Z}_4 \oplus 8\mathbf{Z}$? (6,6,6)

2. (a) Let R be a commutative ring with unity and A be an ideal of R. Show that R/A is a field if and only if A is maximal.

   (b) (i) Determine all ring homomorphisms from **Z** to **Z**.

   (ii) Is the ring 2**Z** isomorphic to 4**Z** ? Justify your answer.

   (c) Let A and B be any two ideals of a ring R. Prove that $\frac{A+B}{A} \cong \frac{B}{A \cap B}$.

   (6,6,6)

3. (a) Define a principal ideal domain. Let F be a field. Prove that F[x] is a principal ideal domain.

P.T.O.

(b) Prove that the ideal $< x >$ is maximal in $\mathbf{Q}[x]$.

(c) Let F be a field and $I = \{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 : a_0, a_1, \ldots, a_n \in F$ and $a_0 + a_1 + \ldots + a_n = 0\}$. Show that I is an ideal of F[x] and find a generator of I. (6,6,6)

4. (a) Define an irreducible polynomial over a field. Let F be a field and $p(x) \in F[x]$. Prove that $<p(x)>$ is a maximal ideal of F[x] if and only if p(x) is irreducible over F.

(b) In a principal ideal domain, prove that an element is irreducible if and only if it is prime.

(c) Prove that $\mathbf{Z}[\sqrt{-3}]$ is not a Euclidean domain. (6½,6½,6½)

5. (a) Let $M_{m\times n}$ (F) denote the vector space of all m×n matrices over a field F. Let $W_1 = \{A \in M_{m\times n}(F) : A_{ij} = 0$ whenever $i > j\}$ and $W_2 = \{A \in M_{m\times n}(F) : A_{ij} = 0$ whenever $i \le j\}$.

Show that W_1 and W_2 are subspaces of $M_{m\times n}$ (F) and that $M_{m\times n}(F) = W_1 \oplus W_2$.

(b) Let V be a vector space over a field F, char F ≠ 2. Let u, v, w be distinct vectors in V. Prove that {u, v, w} is linearly independent if and only if {u+v, v+w, w+u} is linearly independent.

(c) Let $V = \mathbf{R}^4$, For the subspaces $W_1 = \{(x,y,z,t) \in \mathbf{R}^4 : x = t)\}$ and $W_2 = \{(x,y,z,t) \in \mathbf{R}^4 : x = 0, y = -z\}$, find dim W_1, dim W_2, dim (W_1+W_2). (6½,6½,6½)

6. (a) Let V and W be finite dimensional vector spaces over a field F and $T : V \to W$ be a linear transformation. Prove that nullity(T) + rank(T) = dim V.

(b) Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be defined by $T(x_1, x_2) = (x_1 - x_2, x_1, 2x_1 + x_2)$. Let $\beta = \{(1,2), (2,3)\}$ and $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$ be ordered bases for $\mathbf{R}^2$ and $\mathbf{R}^3$ respectively. Compute $[T]_\beta^\gamma$.

(c) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $T(x_1, x_2, x_3) = (3x_1 - 2x_3, x_2, 3x_1 + 4x_2)$. Prove that T is invertible. Further, find $T^{-1}(x_1, x_2, x_3)$. (6½,6½,6½)