[This question paper contains 7 printed pages.]

2137 *Your Roll No.* ....

B.Sc. (Hons.) / III **C**

MATHEMATICS – Paper VIII

(Differential Equations and Mathematical Modeling – II)

Time: 3 Hours *Maximum Marks: 75*

(Write your Roll No. on the top immediately on receipt of this question paper.)

All *Sections are compulsory.*
Attempt any **Two** *parts from each Section.*

SECTION A

1. (a) Describe Lagrange's method of solving a quasi-linear differential equation of the first order. Use it to solve

$$(u^2 - y^2)u_x + x_y u_y + xu = 0 \quad (6)$$

(b) Solve the initial value problem

$u_x + 2u_y = 0$, $u(0, y) = 3e^{-2y}$ using the separation of variables. (6)

P.T.O.

(c) Describe General Integral, Complete Integral and Singular Integral of a first order partial differential equation

$$f(x, y, z, p, q) = 0, \ p = \frac{\partial z}{\partial x}; \ q = \frac{\partial z}{\partial y} \qquad (6)$$

SECTION B

2. (a) Derive the continuity equation

$$\rho_t + \mathrm{div}(\rho\bar{u}) = 0,$$

And Euler's equation of motion

$$\rho(\bar{u}_t + (\bar{u}.\mathrm{grad})\bar{u}) + \mathrm{grad}\ \rho = 0$$

in fluid dynamics. (6.5)

(b) Show that the equation of motion of a long string is $u_{tt} = c^2u_{xx} - g$, where g is the gravitational acceleration. (6.5)

(c) Find the characteristics, characteristic coordinates and then reduce the equation $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} = e^x$ to the canonical form. (6.5)

SECTION C

3. (a) Obtain the D'Alembert solution of the following initial-value problem

$$u_{tt} = c^2 u_{xx}, \ x \in R, \ t > 0$$

$$u(x, 0) = \sin x, \ x \in R,$$

$$u_t(x, 0) = \cos x, \ x \in R.$$ (7)

(b) Find the solution of the characteristic initial-value problem

$$x u_{xx} - x^3 u_{yy} - u_x = 0, \ x \neq 0$$

$$u(x, y) = f(y) \text{ on } y - \frac{x^2}{2} = 0 \text{ for } 0 \leq y \leq 2$$

$$u(x, y) = g(y) \text{ on } y + \frac{x^2}{2} = 4 \text{ for } 2 \leq y \leq 4, \text{ where}$$

$f(2) = g(2)$. (7)

(c) Determine the solution of the initial boundary-value problem :

$$u_{xx} - u_{yy} = 1$$

$$u(x, 0) = \sin(x)$$

$$u_y(x, 0) = x$$ (7)

SECTION D

4. (a) Prove that there exists at most one solution of the wave equation

$$u_{tt} = c^2 u_{xx},\ 0 < x < l,\ t > 0,$$

satisfying the initial conditions

$u(x, 0) = f(x),\ u_t(x, 0) = g(x);\ 0 \leq x \leq l$ and the boundary conditions

$u(0, t) = 0,\ u(l, t) = 0,\ t \geq 0$ where $u(x, t)$ is a twice continuous differentiable function with respect to both x and y. (7)

(b) The Heat conduction problem of a homogeneous rod of length l, where the surface of the rod is insulated to prevent heat loss through the boundary is given by the equation :

$u_t = ku_{xx},\ 0 < x < l,\ t > 0,$

$u(0, t) = 0,\ t \geq 0$

$u(l, t) = 0,\ t \geq 0$

$u(x, 0) = f(x),\ 0 \leq x \leq l.$

Show that the formal series solution of this problem is given as

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right) \quad \text{where}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx \qquad (7)$$

(c) Determine the solution of the initial-boundary value problem :

$u_{tt} - u_{xx} = h,\ 0 < x < 1,\ t > 0$. h is a constant.

$u(x, 0) = x(1 - x),\ 0 \le x \le 1$

$u_t(x, 0) = 0,\qquad 0 \le x \le 1$

$u(0, t) = t,\ u(1, t) = \sin t,\ t \ge 0$ (7)

SECTION E

5. (a) Using Monte Carlo simulation, write an algorithm to calculate the volume of a sphere :

$x^2 + y^2 + z^2 = 1$, that lies in the first octant $x \ge 0,\ y \ge 0,\ z \ge 0$. (6)

(b) Explain linear congruence method for generating random numbers. Does this method have any draw back ? Illustrate with the help of an example. (6)

(c) Solve the Linear Programming Problem :

Minimize : $x_1 - 2x_2$

subject to

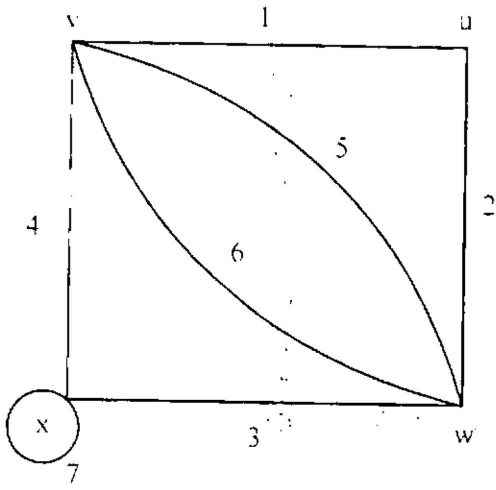
$$3x_1 + 2x_2 \leq 3$$
$$x_1 - x_2 \leq 2$$
$$x_1, x_2 \geq 0. \qquad (6)$$

SECTION F

6. (a) Prove that a connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree. (5)

(b) Prove that if G is an r-regular graph with n vertices, then G has exactly $\frac{nr}{2}$ edges. (5)

(c) Consider the graph G as follows :



Which of the following statement hold for G ? Justify.

(i) Vertices v and x are adjacent.

(ii) Edge 6 is incident with vertex w.

(iii) Vertex x is incident with edge 4.

(iv) Vertex w and edge 5 and 6 form a subgroup of G. (5)