[This question paper contains 4 printed pages.]

2140 *Your Roll No.* ....

B.Sc. (Hons.) III **C**

MATHEMATICS - Paper XI

(Analysis – III)

(Admissions of 2009 and onwards)

Time: 3 Hours *Maximum Marks: 75*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any **TWO** *of* **part (a)** *of each question.*
The **part (b)** *of each question is compulsory.*
Part (a) *carries* **10** *marks and*
Part (b) *carries* **2.5** *marks.*

1. (i) Suppose X is a metric space and S is a subset of X. Prove that

 (a) $cl(S) = S \cup acc(S)$

 (b) $S^{\circ} = S - acc(S^c)$

 (c) $(S^{\circ})^c = cl(S^c)$

 (ii) Suppose $a, b \in \mathbb{R}$ and $a < b$. Show that $\partial([a,b]) = \{a,b\}$.

 (iii) Suppose X is a metric space and Z is a metric subspace of X. Prove that the topology of Z is $\{U \cap Z : U \text{ open in } X\}$.

 (b) Define a bounded metric space. Give example of a bounded and an unbounded metric space.

2. (a) (i) Suppose (X, d) is a metric space and S is a subset of X. Show that S is dense in X if, and only if, for each x in X, there is a sequence in S that converges to x in X.

(ii) Prove that a subset A of $\mathbb{R}$ is totally bounded if, and only if, it is bounded.

(iii) Let A be a nonempty subset of a metric space X. Prove that the function $f(x) = \text{dist}(x, A)$ is continuous.

(b) Suppose X is a metric space and S is a bounded subset of X. Show that closure of S is bounded in X.

3. (a) (i) State Banach's Fixed-Point Theorem. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and there exists $k \in (0, 1)$ such that $|f'(x)| \leq k$ for all $x \in \mathbb{R}$. Then, show that f has a unique fixed point.

(ii) Suppose X is a compact metric space and $f : X \to Y$ is a continuous map. Prove that f(X) is compact and that f is uniformly continuous.

(iii) Suppose T is a subset of $\mathbb{C}$. Show that the corresponding set S of Σ (where Σ is the Riemann sphere) is

(a) a circle if T is a circle, and

(b) a circle minus (0,0,1) if T is a line.

(b) State Liouville's theorem and the Closed Curve Theorem.

4. (a) (i) If f is an entire function and a is any complex number, then prove that f has a power series representation

$f(z) = f(a) + f'(a)(z-a) + (f''(a)/2!)\,(z-a)^2 + \ldots\ldots$ for all $z \in \mathbb{C}$.

(ii) Let $f = u + iv$ be complex function defined in some neighbourhood of a point z. Prove that if f is differentiable at z, then f_x and f_y exist at z and satisfy the Cauchy Riemann equation $f_y = i f_x$.

(iii) Define radius of convergence of a power series. Prove that a power series converges in the interior of its circle of convergence and diverges in the exterior of its circle of convergence where the complex plane is considered as the interior of the circle of infinite radius and a point as a circle of radius zero.

(b) Suppose f is analytic in a region and $f' = 0$ there. Show that f is constant.

5. (a) (i) Find an analytic function f that satisfies the following two conditions.

$f(z+w) = f(z)\,f(w)$ for all z, w in C and $f(x) = \exp(x)$ for all x in $\mathbb{R}$.

(ii) Show that $f^{(k)}(a) = k!/2\pi i \int_C f(w)/(w-a)^{k+1}\,dw$, $k = 1, 2, 3, \ldots$.

Where C is any circle surrounding the point 'a' and f is entire. Hence evaluate $\int_C e^z/z^4\,dz$ where $C : |z| = 2$.

(iii) Prove that the series $\sum_{-\infty}^{\infty} a_n z^n$ is convergent in the domain $D = \{z \mid R_1 < |z| \text{ and } |z| < R_2\}$ where $R_2 = 1/\limsup |a_k|^{1/k}$ and $R_1 = \limsup |a_{-k}|^{1/k}$

Further show that if $R_1 < R_2$, then the sum function $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ is analytic on the annulus D.

(b) Show that an odd entire function has only odd terms in its power series expansion about the origin.

6. (a) (i) If $f(z) = (az + b)/(cz + d)$;

where a, b, c, d are complex numbers and $ad - bc \neq 0$

then prove that f maps circles and lines onto circles and lines.

(ii) Find the Fourier series of the function

$f(x) = x^2/2, \quad -\pi < x < \pi$

Use it to find sum of the series

$1 - 1/4 - 1/9 + 1/16 + 1/25 - \ldots\ldots$

(iii) Let f(x) be a periodic function of period 2π given by

$f(x) = x + x, \quad -\pi < x < \pi$

Obtain its Fourier Coefficients.

(b) Suppose that f is an entire function and $|f(z)| \leq A + B|z|^{3/2}$. Show that f is a linear polynomial.

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