[This question paper contains 7 printed pages.]

2138 *Your Roll No.* ....

B.Sc. (Hons.) / III **C**

MATHEMATICS – Paper IX

(Probability and Statistics)

(Admissions of 2009 and onwards)

Time: 3 Hours *Maximum Marks: 70*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt all questions selecting any two parts from question no.'s 2 to 6

Question No. **1** *is compulsory.*

Each part of Q. **1** *is of* **2** *marks. Each part of the remaining questions is of* **6** *marks.*

Use of scientific calculator is permitted.

1. (i) If X is a random variable such that $E[X] = 3$ and $E[X^2] = 13$. Use Chebyshev inequality to find the lower bound for $P[-2 < X < 8]$.

(ii) Suppose X is a continuous random variable with uniform distribution having mean 1 and variance $\frac{4}{3}$. What is $P[X < 0]$?

(iii) Let X has the pdf $f_x(x) = \begin{cases} cx^3 & 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$

then find the value of c and $p\left[\frac{1}{4} < X < 1\right]$.

(iv) Find the median of the distribution whose pmf is given by

$$p(x) = \begin{cases} \frac{4!}{x!(4-x)!}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{4-x}, & x = 0, 1, 2, 3, 4 \\ 0 & \text{elsewhere} \end{cases}$$

(v) If $X_1, X_2 \ldots X_n$ constitute a random sample of size n from the population given by

$$f(x) = \begin{cases} e^{-(x-\delta)} & \text{for } x > \delta \\ 0 & \text{elsewhere} \end{cases}$$

Then show that the sample mean $\bar{X}$ is biased estimator of δ.

2. (a) A bowl contains 10 chips, of which 8 are marked \$2 each and 2 are marked \$5 each. Let a person choose, at random and without replacement, 3 chips from this bowl. If the person is to receive the sum of the resulting amount, find his expectation.

(b) If $f(x,y) = \begin{cases} e^{-x-y}, & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{elsewhere} \end{cases}$, is the joint pdf of the random variable X, and Y. Find the joint moment generating function of X and Y. Let $Z = X + Y$. Deduce the moment generating function for Z and hence find the mean and variance of Z.

(c) Define the r^{th} cumulant of a random variable X and find k_2, k_3 and k_4 for the exponential distribution.

3. (a) Show that if X is a random variable having the poisson distribution with the parameter λ and $\lambda \to \infty$ then the m.g.f of $Z = \dfrac{X - \lambda}{\sqrt{\lambda}}$ approximate to m.g.f of the standard normal distribution.

(b) Define negative binomial distribution and find its mean and variance.

(c) Let X_1 and X_2 have the joint pdf

$$f(x_1, x_2) = \begin{cases} 6x_2 & 0 < x_2 < x_1 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

verify that

(i) $E[E[X_2/X_1]] = E[X_2]$

(ii) $\text{Var}\,[E[X_2/X_1]] \leq \text{Var}\,[X_2]$

4. (a) Show that the random variable X_1 and X_2 are independent iff the following condition hold

$$P[a < X_1 \leq b,\ c < X_2 \leq d] = P[a < X_1 \leq b]\ P[c < X_2 \leq d]$$

for every $a < b$ and $c < d$, where a, b, c, and d are constants.

(b) If μ and σ are the mean and the standard deviation of a random variable X, then for any positive constants k, show that

$$P\left[|X - \mu| < k\sigma\right] \geq 1 - \frac{1}{k^2},\ \sigma > 0$$

(c) If X and Y have a bivariate normal distribution, then show that the conditional density of Y given X = x has a normal distribution with mean

$$\mu_{Y|x} = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1)$$ and the variance

$$\sigma^2_{Y|x} = \sigma^2_2(1 - \rho^2).$$

5. (a) On any given day Gary is either cheerful (C), so – so (S), or glum (G). If he is cheerful today, then he will be C, S, or G tomorrow with respective probabilities 0.5, 0.4, 0.1. If he is feeling so – so today, then he will be C, S, or G tomorrow with probabilities 0.3, 0.4, 0.3. If he glum today, then he will be C, S, or G tomorrow with probabilities 0.2, 0.3, 0.5. Then find the transition probability matrix, and calculate the probability that, the person who is cheerful today will remain cheerful after two days from today.

(b) Ten competitors in a musical test were ranked by the three judeges A, B and C in the following order :

Rank by A :	1	6	5	10	3	2	4	9	7	8
Rank by B :	3	5	8	4	7	10	2	1	6	9
Rank by C :	6	4	9	8	1	2	3	10	5	7

Using rank correlation method, discuss which pair of judges have the nearest approach to common likings in music.

(c) A paint manufacturer wants to determine the average drying time of a new interior wall paints. If for 12 test areas of equal size he obtained a mean drying time of 66.3 minutes and standard deviation of 8.4 minutes, construct a 95% confidence interval for true mean μ. ($t_{.025} = 20.021$)

6. (a) If $x_1, x_2, \ldots, x_n$ are the values of a random sample from an exponential population. Find the maximum likelihood estomator of its parameter θ.

(b) Show that $\bar{X}$ is a minimum variance unbiased estomator of the mean μ of a normal population.

(c) If $X_1, X_2, \ldots, X_n$ constitute a random sample from an infinite population with mean μ, the variance σ^2 and the m.g.f $M_X(t)$. Then show that, the limiting distribution of $Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$ as $n \to \infty$ is the standard normal distribution.