[This question paper contains 7 printed pages.]

2139 *Your Roll No.* ....

B.Sc. (Hons.) / III **C**

MATHEMATICS – X

(Algebra – III)

(Admissions of 2009 and onwards)

Time: 3 Hours *Maximum Marks: 75*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any **THREE** *parts from questions* **1** *to* **3**, *and any* **TWO** *parts from questions* **4** *to* **6**. *Every part of a question carries* **5** *marks.*

1. (a) Define the kernel of an action of a group G on a set A and show that the kernel of the action of G on A is same as the kernel of the corresponding permutation representation $G \to S_A$. Hence, show that the kernel of G on A is a normal subgroup of G.

P.T.O.

(b) Prove that A_n is generated by the set of all 3-cycles for all $n \geq 3$. Hence or otherwise prove that A_5 is simple.

(c) Let a group G act on a set A. Prove that if a, b$\in$ A and $b = g.a$ for some $g \in G$, then $G_b = gG_ag^{-1}$ where G_a is the stabilizer of a. Deduce that if G acts transitively on A, then the kernel of the group action is $\bigcap_{g \in G}(gG_ag^{-1})$.

(d) Define the class equation of a finite group G. If $|G| = p^2$ for some prime p, then show that G is abelian. Also, show that $G \approx Z_{p^2}$ or $G \approx Z_p \times Z_p$.

OR

For each of S_3 and D_8, compute the centralizers of each element and find the center of each group.

2. (a) Find all subgroups of order 3 in $Z_9 \times Z_3$.

(b) Let $G = \{1, 7, 17, 23, 49, 55, 65, 71\}$ under multiplication modulo 96. Express G as an external and an internal direct product of cyclic subgroups.

(c) Define Composition Series. State the Jordan-Holder theorem. Find all the three composition series of Q_8. Also, list the composition factors.

(d) For a prime p, define a Sylow p-subgroup of a finite group G. Prove that if $|G| = 30$, then either a Sylow 3-subgroup or a Sylow 5-subgroup is normal in G.

OR

(d) If $|G| = 231$, then Z(G) contains a Sylow 11-subgroup of G and a Sylow 7-subgroup is normal in G.

3. (a) Let p(x) be an irreducible polynomial over a field F. If $a \in E$ is a zero of p(x) in some extension E of F, then $F(a) \approx F[x]/\langle p(x)\rangle$.

(b) Prove that $\cos 2\pi/7$ is a root of $8x^3 + 4x^2 - 4x - 1$ and that $2\cos 2\pi/7$ is a root of $x^3 + x^2 - 2x - 1$. Hence show that a regular seven-sided polygon is not constructed with a straightedge and a compass.

(c) Show that the mapping $\Phi : GF(p^n) \to GF(p^n)$ given by $\Phi(a) = a^p$ for all $a \in GF(p^n)$ is a ring-automorphism such that Φ^n is the identity mapping.

(d) Find the splitting field of $x^4 + 1$ over Q.

OR

Show that $Q(\sqrt{3}, \sqrt{5}) = Q(\sqrt{3} + \sqrt{5})$ and find its basis over Q.

(a) State and prove the Cauchy-Schwarz inequality and the triangle inequality on an inner product space V.

(b) Use Gram-Schmidt process to obtain an orthonormal basis for $P_2(R)$ with the standard ordered basis and the inner product $\langle f, g \rangle = \int_{-1}^{1} fg$. Hence compute the orthogonal projection of $f(x) = x^3$ in $P_3(R)$ on $P_2(R)$.

(c) Let T be a linear operator on a finite-dimensional inner product space V. Then prove that the following statements are equivalent.

(i) $TT^* = T^*T = I$.

(ii) $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$.

(iii) $\|T(x)\| = \|x\|$ for all $x \in V$.

OR

(c) Find the minimal solution of the following system of linear equations :

$$x + 2y - z = 4$$

$$x - y + 2z = -11$$

$$x - 5y = 19$$

5. (a) Let T be a linear operator on finite dimensional inner product space V over F. Suppose that the characteristic polynomial of T splits over F. Prove that there exists an orthonormal basis β for V such that $[T]_\beta$ is upper triangular.

(b) Let T be a linear operator on a finite-dimensional complex inner product space V. Then prove that T is normal if and only if there exists an orthonormal basis for V consisting of eigenvectors of T.

(c) Let A be a n×n real matrix. Prove that A is self-adjoint if and only if A is orthogonally equivalent to a real diagonal matrix.

OR

(c) Let T be a self-adjoint operator on a finite-dimensional inner product space V. Prove that for all x, y ∈ V

$$\|T(x) \pm ix\|^2 = \|T(x)\|^2 + \|x\|^2.$$

Deduce that T − iI is invertible and that $(T + iI)(T - iI)^{-1}$ is unitary.

6. (a) Let V be a finite-dimensional vector space over the field F and α any non-zero vector in V. Let T be a linear operator on V. Prove that

$$\{\alpha, T\alpha, T^2\alpha, \ldots, T^{k-1}\alpha\}$$

is a basis for the T-cyclic subspace Z(α; T), where k is the degree of the T-annihilator of α. Also find the minimal polynomial of the linear operator U on Z(α; T) induced by T.

(b) Find the minimal polynomial and the rational form of the following matrix :

$$\begin{bmatrix} c & 0 & -1 \\ 0 & c & 1 \\ 1 & 1 & c \end{bmatrix}$$

(c) Define Jordan form of a square matrix. If A is a complex 5 × 5 matrix with characteristic polynomial

$$f(x) = (x-2)^3(x+7)^2$$

Find all the possible Jordan forms for A.

OR

(c) State cyclic decomposition theorem. Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Let W be the null space of T – 2I. Prove that W has no complimentary T-invariant subspace.