This question paper contains 7 printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **8803**

Unique Paper Code : **235104** : **C**

Name of the Paper : **I.3 : Algebra–I**

Name of the Course : **B.Sc. (H) Mathematics (Part I)**

Semester : **I**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

All six questions are compulsory.

Do any *two* parts from each question.

1. (*a*) Let u, v, w be the zeros of the cubic polynomial $4t^3 - 7t^2 - 3t + 2$. Determine a cubic polynomial whose zeros are $u - \frac{1}{vw}, v - \frac{1}{uw}, w - \frac{1}{uv}$. 6

(*b*) (*i*) Prove that the equation $t^7 - 5t^5 - 4t + 3 = 0$ cannot have more than two positive roots. Show that it has at least *four* complex roots.

(*ii*) Estimate an upper bound for the real zeros of

$$p(t) = t^7 - t^6 + t^5 + 2t^4 - 3t^3 + 4t^2 + t - 2.$$

State the result used. 3,3

P.T.O.

(c) (i) Consider the polynomial equation

$$t^4 + pt^3 + qt^2 + rt + s = 0.$$

Prove that if the sum of two of its roots is equal to the sum of the other two, then $p^3 + 8r = 4pq$.

(ii) Find the polar representation for the complex number

$$z = 1 - \cos a + i \sin a, a \in [0, 2\pi).$$ 3,3

2\. (a) Solve the equation :

$$z^4 = 5(z-1)(z^2 - z + 1)$$ 6.5

(b) (i) Compute :

$$z = \frac{(1-i)^{10}(\sqrt{3}+i)^5}{(-1-i\sqrt{3})^{10}}.$$

(ii) Let z_1, z_2, z_3 be the coordinates of vertices A, B, C of a triangle. If $w_1 = z_1 - z_2$ and $w_2 = z_3 - z_1$, prove that $\hat{A} = 90^0$ iff $\text{Re}(w_1 \cdot \overline{w}_2) = 0$. 3,3.5

(c) (i) Find all complex numbers z such that $|z| = 1$ and $\left|\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right| = 1$.

(*ii*) Consider three distinct points $M_1(z_1), M_2(z_2)$ and $M_3(z_3)$, where $z_1 = 4+3i$, $z_2 = 4+7i, z_3 = 8+7i$.

Find the measure of the oriented angles $M_3\hat{M}_1M_2$ and $M_2\hat{M}_1M_3$. 3.5,3

3. (*a*) Let $a, b \in$ **Z**. Define $a \sim b$ if $a \equiv b \pmod{n}$.

(*i*) Prove that $\sim$ defines an equivalence relation on Z.

(*ii*) If $n = 7$, find the equivalence classes of 4 and -4.

(*iii*) What is the quotient set determined by this equivalence relation for $n = 7$? 2,2,1

(*b*) (*i*) Define floor and ceiling function on **R**. At what values of x do the jumps in the graph of $y = |2x - 3|$ occur ? 3

(*ii*) Show that $|\mathbf{Z}| = \aleph_0$. 2

(*c*) (*i*) Show that the functions $f : \mathbf{R} \to (1, \infty)$ and $g : (1, \infty) \to \mathbf{R}$ defined by $f(x) = 3^{2x} + 1$, $g(x) = \frac{1}{2}\log_3(x-1)$ are inverses.

(*ii*) Let ~ define an equivalence relation on a set A and $a, b \in A$. Prove that $a \sim b$ if and only if $[a] = [b]$. 3,2

4. (*a*) Prove that the intervals (3, 5) and (9, ∞) have the same cardinality. 5

(*b*) Let $(a, b) \sim (c, d)$ in $\mathbf{R}^2$ if and only if $ab = cd$. Prove that ~ defines an equivalence relation on $\mathbf{R}^2$. Give the geometrical interpretations of the equivalence classes of (0, 0) and (1, 1). 5

(*c*) (*i*) Show that the function $f : A \to \mathbf{R}$ defined by $f(x) = 5 - \frac{1}{1+x}$, where $A = \{x \in \mathbf{R} / x \neq -1\}$ is one to one. Find the range of f and a suitable inverse.

(*ii*) Let A and B be non-empty sets. Prove that $A \times B = B \times A$, if and only if $A = B$. 4,1

5. (*a*) Determine the general solution of the following system in parametric vector form :

$$\begin{aligned} x-2y+9z+5w &= 4 \\ x-y+6z+5w &= -3 \\ -2x-6z+w &= -2 \\ 4x+y+9z+w &= -9 \end{aligned}$$

7.5

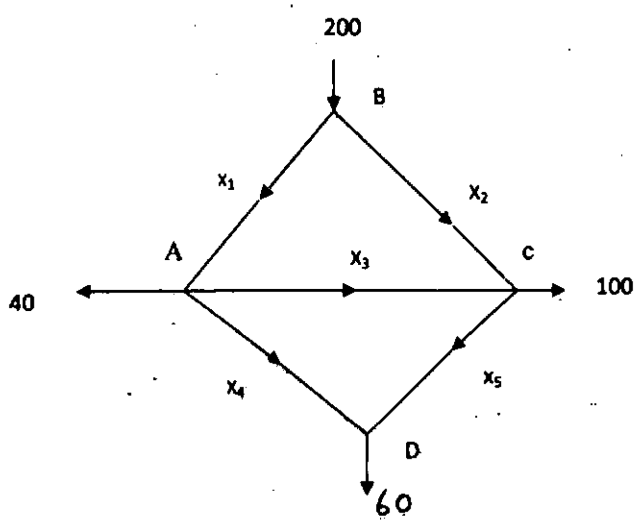
(*b*) (*i*) Let T be a linear transformation whose standard matrix is :

$$A = \begin{bmatrix} -5 & 10 & -5 & 4 \\ 8 & 3 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix}.$$

Is T a one-to-one mapping ? Does T maps R^4 onto R^4 ?

(*ii*) Define subspace of R^n. Prove that the null space of an $n \times n$ matrix A is a subspace of R^n. 4,3.5

(*c*) Find the general traffic pattern in the freeway network shown in the figure below. (Flow rates are in cars/minute). Describe the general traffic pattern when the road whose flow is x_4 is closed. 7.5



6. (*a*) (*i*) Define basis and dimension of a subspace H of R^n. Is

$$H = \{(a,b,c,d) / a = b + c + d\}$$

a subspace of R^4 ? Justify.

(*ii*) Let $T : R^2 \to R^3$ be a linear transformation such that :

$$T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2)$$

Find x such that $T(x) = (-1,4,9)$. Also find the standard matrix of T. 4, 3.5

(*b*) Find the bases and dimensions for Col A and Nul A where

$$A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix}.$$

Hence find the rank of A. 7.5

(c) (i) Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Does $\{v_1, v_2, v_3\}$ span R^4 ? Give reasons.

(ii) Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T maps R^2 onto R^3 ? 3.5,4