This question paper contains 2 printed pages.

Your Roll No. ....

Sl. No. of Ques. Paper	**: 1354**	**F–7**
Unique Paper Code	**: 2351501**	
Name of Paper	**: Algebra – IV (Group Theory II)**	
Name of Course	**: B.Sc. (Hons) Mathematics (Erstwhile FYUP)**	
Semester	**: V**	
Duration :	**: 3 hours**	
Maximum Marks	**: 75**	

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any two parts from each question. All questions are compulsory.

1. (a) Define Inn (G) and prove that Inn (G) $\leq$ Aut (G), G is a group.

   (b) Find Aut (Z_8) and also make its Cayley table.

   (c) (i) Find Aut (Z).

   (ii) If G is non- Abelian group, then show that Aut (G) cannot be cyclic.

   (6.5, 6.5, 6.5)

2. (a) Prove that commutator subgroup G′ of a group G is a characteristic subgroup of G.

   (b) (i) Show that $Z_8 \oplus Z_2$ is not isomorphic to $Z_4 \oplus Z_4$.

   (ii) Let H is a subgroup of a group G be such that it contains commutator subgroup G′ of G, then prove that H is normal subgroup of G.

   (c) Let G and H be finite cyclic groups. Prove that G $\oplus$ H is cyclic if and only if |G| and |H| are relatively prime. (6, 6, 6)

3. (a) Show that if G is the internal direct product of $H_1, H_2, \ldots, H_n$ and $i \neq j$ with $1 \leq i, j \leq n$, then $H_i \cap H_j = \{e\}$.

   (b) Express $U(110)$ as

   (i) External direct product of cyclic additive groups of the form $\mathbb{Z}_n$

   (ii) Internal direct product of its proper subgroups.

   (c) Show that there are two abelian groups of order 108 that have exactly four subgroups of order 3. (6, 6, 6)

P.T.O.

4. (a) Let G be a group. Prove that the mapping $\varphi : G \times G - G$, defined by $\varphi(g, a) = g.a = gag^{-1}$ is a group action.
Is this action a trivial action if G is Abelian? Find its kernel and stabilizer G_a.

(b) If $G = D_{10}$, the *Dihedral group* of order 10 and $A = \{1, r, r^2, r^3, r^4\}$ is a subgroup of G, then show that $C_G(A) = A$ and $N_G(A) = G$.

(c) Let G be a group acting on a non-empty set A. Prove that the relation on A defined by

$$a \sim b \text{ if and only if } a = g \cdot b \text{ for some } g \in G$$

is an equivalence relation. Also prove that for each $a \in A$, the number of elements in the equivalence class containing a is $|G: G_a|$ i.e. the index of the stabilizer of a in G. (6.5, 6.5, 6.5)

5. (a) Write the conjugate class equation for a finite group G. Let P be a group of prime power order p^α, $\alpha \geq 1$. Use class equation to prove $|\mathbb{Z}(P)| > 1$.

(b) Let $\sigma_1 = (1\ 2\ 3\ 4\ 5)$ *and* $\sigma_2 = (1\ 3\ 2\ 4\ 5)$. Is σ_1 conjugate to σ_2 in S_5. Are these conjugate in A_5 as well? Explain.

(c) Find all the conjugacy classes and their sizes of D_8, the group of symmetries of a square. (6, 6, 6)

6. (a) State sylow's first theorem. Exhibit all sylow 3 - subgroups and sylow 2 - subgroups of A_4.

(b) Let $|G| = 56$, G be a group. Prove that either sylow 2 - subgroup or sylow 7 - subgroup is unique.

(c) Is A_n simple $\forall\ n \geq 5$? Prove that A_n, $n \geq 5$ cannot have a proper subgroup of index $< n$. (6.5, 6.5, 6.5)