[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 1113 **G** **Your Roll No................**

Unique Paper Code : 235104

Name of the Paper : Algebra – I

Name of the Course : **B.Sc. (Hons.) Mathematics credit Course -I**

Semester : I

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Do any **two** parts from each questions.

1. (a) Find a cubic equation whose roots are the squares of the roots of the equation

$x^3 - x^2 + 3x - 10 = 0.$ (6)

(b) State Descartes' rule of signs. Using this rule verify that (6)

$$t^{11} + t^8 - 3t^5 + t^4 + t^3 - 2t^2 + t - 2$$

has at most 5 positive and 2 negative zeros. Deduce that it has at least 4 non-real zeros.

(i) Consider the polynomial equation $x^4 + px^3 + qx^2 + rx + s = 0$.

Prove that if the product of two of its roots is equal to the product of the other two, then $r^2 = p^2s$. (4)

(ii) Let z_1, z_2, z_3 be non-zero complex coordinates of the vertices of the triangle $A_1A_2A_3$. If $z_1^2 = z_2z_3$ and $z_2^2 = z_1z_3$, show that triangle $A_1A_2A_3$ is equilateral. (2)

2. (a) (i) Find the polar representation of the number $z = 2 + 2i$. (2)

(ii) Prove that (4½)

$$\sin 5t = 16 \sin^5 t - 20 \sin^3 t + 5 \sin t;$$

$$\cos 5t = 16 \cos^5 t - 20 \cos^3 t + 5 \cos t.$$

(b) Solve the equation : (6½)

$$z^7 - 2iz^4 - iz^3 - 2 = 0.$$

(c) On the sides AB, BC, CD, DA of quadrilateral ABCD and exterior to the quadrilateral, we construct squares of centers O_1, O_2, O_3 and O_4 respectively. Prove that O_1O_3 is perpendicular to O_2O_4 and $O_1O_3 = O_2O_4$. (6½)

3. (a) For $a, b \in N$, define $a \sim b$ if and only if $a^2 + b$ is even.

(i) Prove that '$\sim$' defines an equivalence relation on N.

(ii) What are the equivalence classes of 0 and 1 ?

(iii) Find the quotient set determined by this equivalence relation ? (5)

(b) Let '$\sim$' denote an equivalence relation on a set A, let $a \in A$, then for any $x \in A$, prove that $x \sim a$ if and only if $\bar{x} = \bar{a}$. (5)

(c) Let $n > 1$ be a fixed natural number, prove that congruence mod n is an equivalence relation on Z. (5)

4. (a) Define $f: Z \to Z$ by $f(x) = 2x^2 + 7x$. Determine whether f is one-to-one and\or onto. (5)

(b) Define Countable set. Show that intervals (0,1) and (3,5) have the same cardinality. (5)

(c) Prove, using Principle of Mathematical Induction that every integer greater than 1 is a prime or a product of primes. (5)

5. (a) Write the following system as a vector equation and as a matrix equation. Row reduce the augmented matrix into reduced echelon form. Describe the general solution in parametric form.

$$x_1 + 3x_2 + x_3 = 1$$

$$-4x_1 - 9x_2 + 2x_3 = -1$$

$$-3x_2 - 6x_3 = -3$$

(7½)

(b) Boron sulphide reacts violently with water to form boric acid and hydrogen sulphide gas (the smell of rotten egg). The unbalanced equation is

$$B_2S_3 + H_2O \rightarrow H_3BO_3 + H_3S$$

For each compound, construct a vector that lists the numbers of atoms of boron, sulfur, hydrogen, and oxygen. Balance the chemical equation using vector equation approach. (7½)

(c) (i) Find the value(s) of h for which the vectors are linearly dependent. Justify your answer.

$$\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$$

(ii) Give a geometric description of Span $\{v_1, v_2\}$ for the vectors

$$v_1 = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 12 \\ 3 \\ -9 \end{bmatrix}$$

(4½,3)

6. (a) (i) Let $T: R^2 \rightarrow R^2$ be a linear transformation, $T(e_1) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Find the images of $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(ii) Show that the transformation T defined by $T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$ is not linear. (4½,3)

(b) (i) Let $T: R^2 \rightarrow R^2$ be a linear transformation which rotates points (about the origin) through $-\pi/4$ radians clockwise. Find the standard matrix of T.

(ii) The vector x is in a subspace H with a basis $B = \{b_1, b_2\}$. Find $[x]_B$, the B-coordinate vector of x where

$$b_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -2 \\ 7 \end{bmatrix}, \quad x = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$ (4½,3)

(c) Find bases for Col A and Nul A. Hence find rank of A.

$$A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix}$$ (7½)