[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 1118 **G** **Your Roll No...............**

Unique Paper Code : 235503

Name of the Paper : Analysis – IV (MAHT-502)

Name of the Course : **B.Sc. (Hons.) Mathematics**

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question.

1. (a) Let d_1 and d_2 be two metrics on a non-empty set X. Let

$$d(x, y) = \sqrt{d_1^2(x, y) + d_2^2(x, y)}$$

$$d^*(x, y) = \max(d_1(x, y), d_2(x, y)) \text{ for all } x, y \in X.$$

What is the relation between these two metrics ?

(b) Suppose (x, d) is a metric space, Z is a set and $f : Z \to X$ is an injection function.

Show that $(a,b) \to d(f(a), f(b))$ is a metric on Z.

(c) Suppose X is a metric space, $z \in X$ and S is a subset of X. Show that $z \in acc(S)$ if, and only if, $z \notin iso(S)$ and $dist(z, S) = 0$.

2. (a) Suppose X is a metric space, $x \in X$ and S is a subset of X. Show that $x \in$ boundary (S) if and only if, every open ball of X centred at x has non –empty intersection with both S and S^c.

(b) What is an isometry ? Show that an isometry from a metric space X onto a metric space (Y, d_y) is a homeomorphism.

(c) Suppose (X, d) is a metric space, $w \in X$ and A is a subset of X. Then, show that

dist (w, cl (A)) = dist(w, A),

where cl(A) is the closure of A.

3. (a) Suppose X is a metric space and $S \subseteq X$. Prove that

(i) $(S^0)^c = Cl(S^c)$

(ii) $S^0 = \{x \in X: dist(x, S^c) > 0\}$

(b) Suppose X is a metric space and S is a non-empty subset of X. Then prove that

$$diam(cl(S)) = diam(S).$$

Do we also have $diam(int(S)) = diam(S)$? Justify your answer.

(c) Suppose (X, d) is a metric space and Z is a metric subspace of X. Show that the collection of open subsets of Z is $\{U \cap Z \mid U \text{ is open in } X\}$.

4. (a) Suppose X is a metric space and $S \subseteq X$. Prove that

   (i) $\partial(\partial S) \subseteq \partial S$.

   (ii) $\bar{S}$ is the smallest superset of S that is closed in X.

   (b) Let A be a nonempty subset of a metric space X. Show that $z \in cl(A)$ if and only if, there is a sequence in A that converges to z in X.

   (c) If $x = \langle x_n \rangle$ has a subsequence which converges to z. Show that

$$\text{dist}(z, \{x_n : n \in N\}) = 0.$$

   Give an example to show that converse to above statement is not necessarily true.

5. (a) Suppose X is a metric space and S is a bounded subset of X,

   (i) Show that closure of S is bounded in X.

   (ii) Is S also a totally bounded ? Justify your answer.

   (b) Let $f: X \to Y$ be a function from the metric space X into other metric space. Show that f is a continuous at every isolated point of X.

   (c) Let $f: (X, d) \to (Y, e)$ be uniformly continuous. Prove that the sequence $\langle x_n \rangle$ is Cauchy in X if and only if the sequence $\langle f(x_n) \rangle$ is Cauchy in Y.

6. (a) State Banach contraction Property. Give an example of a complete metric space X and a function $f: X \to X$ such that $d(f(x), f(y)) <= d(x,y)$ for all $x, y \in X$, but f has no fixed point in X.

(b) Define a connected space. Prove that a metric space X is connected if and only if, every continuous function $f : X \to \{0, 1\}$ (a discrete space) is a constant function.

(c) Define a compact metric space. Prove that the continuous image of a compact metric space is compact.