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Your Roll No. ....

1466

B.Sc. (Hons.)/II **A**

MATHEMATICS—Paper VII

(Algebra-II)

(Admissions of 2009 and onwards)

Time : 3 Hours *Maximum Marks : 75*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt *All* questions.

1. (*a*) Let H and K be subgroups of a finite group G. Prove that :

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$ 6½

(*b*) Show that the commutator subgroup G' of a group G is a normal subgroup of G. 6

Or

(*a*) Prove that every subgroup of D_n of odd order is cyclic. 4

(*b*) Show that for :

$$n \geq 3, Z(S_n) = \{I\}.$$ 4½

(*c*) Let H be a subgroup of G. If index of H in G is 2, show that H is normal in G. 4

P.T.O.

2. (*a*) Let G be a finite abelian group and let p be a prime that divides the order of G. Show that G has an element of order p. 6

(*b*) Prove that there is no homomorphism from :

$$Z_{16} \oplus Z_2 \text{ onto } Z_4 \oplus Z_4.$$ 6½

Or

(*a*) Show that a group of order 35 is cyclic by using N/C theorem. 6

(*b*) Show that :

$$\frac{G}{Z(G)} \cong 2nn(G)$$

where Z(G) denotes the centre of G. Deduce that Aut G is cyclic implies G is abelian. 6½

3. (*a*) Let R be a commutative ring with unity and let A be an ideal of R. Show that $\frac{R}{A}$ is a field if and only if A is maximal. 6

(*b*) Show that $I = \langle 2 + 2i \rangle$ is not a prime ideal of $Z[i]$. How many elements are in $\frac{Z[i]}{I}$? 6½

Or

(*a*) Show that if m and n are distinct positive integers, then the ring mz is not isomorphic to the ring nz. 6

(*b*) Determine all ring homomorphisms from Z_{12} to Z_{30}. 6½

4. (*a*) Show that in a principal ideal domain an element is irreducible if and only if it is prime. 6½

(*b*) (*i*) Let D be a Euclidean domain and d the associated function. Show that u is a unit in D, if and only if $d(u) = d(1)$. 4

(*ii*) Determine the units in $Z[i]$. 2

Or

(*a*) Assuming ascending chain condition, show that every principal ideal domain is a unique factorization domain. 6½

(*b*) Show that $x^4 + 1$ is irreducible over Q but reducible over Z_p for every prime p. 6

5. (*a*) State and prove Replacement theorem. 6

(*b*) Let V, W and Z be vector spaces and let $T : V \rightarrow W$ and $U : W \rightarrow Z$ be linear. Prove that UT is one-to-one, then T is one-to-one. Must U also be one-to-one ? Justify your answer. 6½

Or

(*a*) Let u, v and w be distinct vectors of a vector space v. Show that $\{u, v, w\}$ is a basis for v if and only if $\{u + v, v + w, w + u\}$ is a basis for v. 6

(*b*) Let v and w be finite dimensional vector spaces with ordered bases β and γ. Let $T : V \to W$ be linear. Show that T is invertible if and only if $[T]_\beta^\gamma$ is invertible. 6½

6. (*a*) Prove that if W is a subspace of a finite dimensional vector space v, then $\dim w + \dim w^\circ = \dim v$. 6

(*b*) Let T be the linear operator on $M_{2\times 2}$ (R) defined by $T(A) = A^t$. Show that ± 1 are only eigen values of T and T is diagonalizable. 6½

Or

(*a*) Suppose that W is a finite dimensional vector space and $T : V \to W$ is linear. Prove that :

$$N(T^t) = (R(T))^\circ.$$ 6

(*b*) Show that the characteristic polynomial of any diagonalizable operator splits but the converse need not be true. 6½