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Your Roll No.................

1464

B.Sc. (Hons.)/II **A**

MATHEMATICS—Paper V

(Analysis–2)

(Admissions of 2009 and onwards)

Time : 3 Hours *Maximum Marks : 75*

(*Write your Roll No. on the top immediately on receipt of this question paper.*)

Attempt *All* questions. Internal choice is given. Marks for each question and distribution of marks for different parts of every question are indicated.

1. (*a*) Show that a bounded function f on the closed interval $[a, b]$ is integrable if and only if for each $\epsilon > 0$, there exists a partition P of $[a, b]$ such that :

$$U(f, P) - L(f, P) < \epsilon.$$ 7

(*b*) Suppose the functions f and g are integrable on $[a, b]$ and suppose :

$$f(x) \leq g(x) \text{ for all } x \in [a, b].$$

Show that :

$$\int_a^b f \leq \int_a^b g.$$ 5

Or

(*a*) Show that a continuous function f on the closed interval $[a, b]$ is integrable on $[a, b]$. 5

(*b*) Let f be a bounded function on $[a, b]$ so that there exists $B > 0$ such that :

$$|f(x)| \leq B \text{ for all } x \in [a,b].$$

Show that :

$$U(f^2, P) - L(f^2, P) \leq 2B\,[U(f, P) - L(f, P)]$$

for all partitions P of $[a, b]$.

Hence show that if f is integrable on $[a, b]$ then f^2 is integrable on $[a, b]$. 7

2. (*a*) Let $\langle f_n \rangle$ be a sequence of functions on $A \subseteq \mathbf{R}$ to $\mathbf{R}$. Let $f : A \to \mathbf{R}$ be another function. What is meant by saying that the sequence $\langle f_n \rangle$ converges :

(*i*) pointwise to f on A,

(*ii*) uniformly to f on A.

Let $f_n(x) = \dfrac{x}{n}$, $x \in \mathbf{R}$, for each $n \in \mathbf{N}$.

Find if $<f_n>$ is :

(*i*) pointwise convergent on $(0, \infty)$

(*ii*) uniformly convergent on $[-2, 1]$.

Justify your answers. 6

(*b*) Let $\sum_{n=1}^{\infty} f_n$ be a series of real functions on $A \subseteq \mathbf{R}$ to $\mathbf{R}$.

Let $<M_n>$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} M_n$ is convergent, and for each n,

$$|f_n(x)| \leq M_n \text{ for all } x \in A.$$

Show that $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on A.

Use this result to show that :

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 + n^2}$$

is uniformly convergent on $\mathbf{R}$. 6

(*c*) Show that the exponential function is strictly increasing on $\mathbf{R}$ and has range equal to :

$$\mathbf{R}^+ = \{y \in \mathbf{R} : y > 0\}.$$ 6

Or

(*a*) Let $\langle f_n \rangle$ be a sequence of continuous real functions on a set $A \subseteq R$.

Let $f : A \to R$ be another function such that the sequence $\langle f_n \rangle$ converges uniformly to f on A. Show that f is continuous on A. Does the result apply to the sequence $\langle f_n \rangle$ with :

$$f_n(x) = x^n, \ x \in [0, 1], \ n \in \mathbf{N} \ ?$$

Justify your answer. 8

(*b*) Check uniform convergence of the sequence $\langle E_n \rangle$ on any interval $[-A, A]$, $A > 0$ where $E_n : \mathbf{R} \to \mathbf{R}$ is :

$$E_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots + \frac{x^n}{n!},$$

$x \in \mathbf{R}, n \in \mathbf{N}$. 5

(*c*) Let for each :

$$n \in \mathbf{N}, \ g_n : [0, \ \infty) \to \mathbf{R}$$

be given by :

$$g_n(x) = \frac{e^{-nx}}{n} \ , \ \ x \geq 0;$$

and g_n' be the derivative of g_n.

Find the functions :

(i) $\left(\lim_{n\to\infty} g_n\right)$

(ii) $\lim_{n\to\infty} \left(g_n'\right)$.

Find if the two limits above are uniform on $[0, \infty)$.

Are the two functions equal ? 5

3. (a) Define the radius of convergence R of a power series :

$$\sum_{n=0}^{\infty} a_n x^n$$

If R > 0 and K is a closed and bounded interval contained in (–R, R), then show that the power series is uniformly convergent on K. 5

(b) Define absolute convergence of the improper integral :

$$\int_a^{\infty} f(t)\, dt.$$

Show that :

$$\int_1^{\infty} \frac{\cos t}{t}\, dt$$

is not absolutely convergent. 5

Or

(*a*) Find the radius of convergence of the series : 2

$$\sum_{n=2}^{\infty} \frac{1}{\log n} x^n.$$

(*b*) Show by integrating the series for $\frac{1}{1+x}$ that if $|x| < 1$, then : 4

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, |x| < 1.$$

(*c*) Define the Beta function β.

Show that for :

$$p, q \in \mathbf{R}, p > 0, q > 0.$$

$$\beta(p, q) = 2 \int_{0^+}^{(\pi/2)^-} (\sin u)^{2p-1} (\cos u)^{2q-1} \, du.$$

Deduce that $\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$. 4

4. (*a*) Evaluate :

$$\iint_D \log(x^2 + y^2) \, dx \, dy$$

where D is the region in the first quadrant lying between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Also plot the region D. 6

(*b*) Find the centre of mass of hemispherical region W defined by :

$$x^2 + y^2 + z^2 \leq 1, \ z \geq 0.$$

Assume that the density is constant. 6

Or

(*a*) Evaluate :

$$\iint_{D_a} e^{-(x^2 + y^2)} dx\, dy$$

where D_a is the disc $x^2 + y^2 \leq a^2$.

Further show that the Gaussian integral :

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$ 7

(*b*) Let W be the region bounded by the planes $x = 0$, $y = 0$ and $z = 2$ and the surface $z = x^2 + y^2$ and lying in the quadrant $x \geq 0, y \geq 0$. Compute :

$$\iiint_W x\, dx\, dy\, dz.$$ 5

5. (a) Find the line integral of the vector field :

$$\vec{F} = y\hat{i} - x\hat{j} + \hat{k}$$

along the path :

$$\vec{C}(t) = (\cos t)\,\hat{i} + (\sin t)\,\hat{j} + \frac{t}{2\pi}\hat{k};\ 0 \le t \le 2\pi,$$

joining the points (1, 0, 0) and (1, 0, 1). 4

(b) Let :

$$\vec{\Phi} : \mathbf{R}^2 \to \mathbf{R}^3$$

be given by :

$$x = u\cos v,\ y = u\sin v,\ z = u;$$

where $u \ge 0$.

Find the tangent plane at $\vec{\Phi}(1, 0)$. 4

(c) Find $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$, where S is the surface :

$$x^2 + y^2 + 3z^2 = 1,\ z \le 0 \text{ and}$$

$$\vec{F} = y\hat{i} + x\hat{j} + 2x^3y^2\,\hat{k}.$$ 4

Or

(*a*) Evaluate $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$, where :

$$\vec{F}(x, y, z) = \sin z\,\hat{i} + \cos\sqrt{y}\,\hat{j} + x^3\,\hat{k};$$

and C is the line segment from (1, 0, 0) to (0, 0, 3). 4

(*b*) Find the work done by $\vec{F}$ along the circle of radius a in the Y-Z plane, where the force field $\vec{F}$ is given by

$$\vec{F}(x, y, z) = x^3\,\hat{i} + y\hat{j} + z\hat{k}. \quad 4$$

(*c*) Find the area of the graph of the function :

$$f(x, y) = \frac{2}{3}(x^{3/2} + y^{3/2})$$

over the domain $D = [0, 1] \times [0, 1]$. 4

6. (*a*) State Green's theorem. Use it to derive a formula for the area of a region D in $\mathbf{R}^2$, bounded by a curve C. Verify this area formula if D is the disc $x^2 + y^2 \leq r^2$. 6

(*b*) Find the integral of :

$$\vec{F}(x, y, z) = z\hat{i} - x\hat{j} - y\hat{k}$$

around the triangle with vertices (0, 0, 0), (0, 2, 0) and (0, 0, 2) using Stokes' theorem. 5

Or

(*a*) Evaluate :

$$\int_{C} (2x^3 - y^3)\, dx + (x^3 + y^3)\, dy,$$

where C is the unit circle and verify Green's theorem for this case. 6

(*b*) State Gauss divergence theorem. Use it to evaluate :

$$\iint_{S} \vec{F} \cdot d\vec{S},$$

where $\vec{F} = 2x\hat{i} + y^2\hat{j} + z^2\hat{k}$; and S is the unit sphere $x^2 + y^2 + z^2 = 1$. 5