This question paper contains 4 printed pages.]

9658 *Your Roll No.* ....

B.A. / B.Sc. (Hons.) / II **B**

MATHEMATICS – Unit IX

(Analysis – III)

(Admissions of 2008 and before)

Time : 2 Hours *Maximum Marks : 38*

(Write your Roll No. on the top immediately on receipt of this question paper.)

*Attempt any **four** parts from Question No. 1. Attempt any **two** parts from Q. No. 2 to Q. No. 5.*

1. (a) Prove that in a metric space (X, d), each closed sphere is a closed set. 2½

(b) Let (X, d) and (Y, e) be two metric spaces. Prove that a sequence $\langle (x_n, y_n) \rangle$ in the product space converges to (*x*, *y*) if and only if $x_n \to x$ and $y_n \to y$. 2½

(c) What is an isometry ? Prove that every isometry from a metric space X onto a metric space Y is a homeomorphism. 2½

(d) Prove that every contraction mapping on a metric space is uniformly continuous. 2½

(e) Define completion of a metric space. Discuss the completion of Q (the set of rationals) as R (the set of all reals). 2½

2. (a) Let (X, d) be a metric space. Define $\rho(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$ $\forall\, x, y \in X$. Prove the ρ is a metric on X. Also prove that the metrics d and ρ are equivalent. 3½

(b) Let (X, d) be a metric space and A, B ⊆ X. Then prove that

(i) A° is the largest open subset of A.

(ii) $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$

(iii) $(A \cup B)^\circ \neq A^\circ \cup B^\circ$

[Here, A° denotes the interior of set A]. 3½

(c) Show that the function $f : R \to]-1, 1[$ defined by $f(x) = \dfrac{x}{1 + |x|}$ is a homeomorphism. 3½

3. (a) Show that a closed subset of a compact metric space is compact. Also show that a compact subset of a metric space is closed. 3½

(b) Let (X, d) be a metric space and let A, B be closed subsets of X such that $A \cup B$ and $A \cap B$ are connected. Show that both A and B are also connected. 3½

(c) Show that every closed interval [a, b] is compact. 3½

4. (a) State and prove Banach's fixed point theorem. 3½

(b) Show that a complete subset of a metric space is closed. Also show that a closed subset of a complete metric space is complete. 3½

(c) Prove that the space of all continuous real-valued functions on [0,1] with metric d defined by

$$d(f, g) = \sup \{|f(x) - g(x)| : x \in [0,1]\}$$

is a complete metric space. Give an example of an incomplete metric on this space. 3½

5. (a) State and prove Schwarz's theorem. 3½

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y)=\begin{cases} x^2\tan^{-1}\left(\dfrac{y}{x}\right)-y^2\tan^{-1}\left(\dfrac{x}{y}\right) \\ 0, \quad \text{elsewhere} \end{cases}$$

if $x \neq 0, y \neq 0$

Show that $f_{yx}(0,0) \neq f_{xy}(0,0)$. 3½

(c) Find the maxima and minima of the function $f(x,y)=x^4+y^4-6(x^2+y^2)+8xy$. 3½