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Your Roll No. ....

9654

B.A./B.Sc. (Hons.)/II **B**

MATHEMATICS—Unit V

(Algebra—II)

(Admissions of 2008 and before)

Time : 2 Hours *Maximum Marks : 38*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *one* question from each Section.

Section I

1. (*a*) Write all the right cosets of H in G, where G = $\langle a \rangle$ is a cyclic group of order 10 and H = $\langle a^2 \rangle$ is a subgroup of G. 4

(*b*) Let G be a group and G' be the commutator subgroup of G. Prove G' $\leq$ G. Further, show that if G is a non-abelian simple group, then G' = G. 6

2. (*a*) If H and K are subgroups of G, $o(H) > \sqrt{o(G)}$ and $o(K) > \sqrt{o(G)}$, then prove $H \cap K \neq \{e\}$. Hence show if $o(G) = pq$, where p and q are prime numbers, $p > q$, then G cannot contain more than one subgroup of order p. 6

(*b*) If a cyclic subgroup T of G is normal in G, then show that every subgroup of T is normal in G. 4

Section II

3. (*a*) Let G be the group of all non-zero complex numbers under multiplication and let $\bar{G}$ be the group of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where a and b are not both zero, under matrix multiplication. Show that G and $\bar{G}$ are isomorphic by exhibiting an isomorphism of G onto $\bar{G}$. 5

(*b*) If G is a group, H a subgroup of G and S is the set of all right cosets of H in G, then prove there is a homomorphism θ of G into A(S). 5

4. (*a*) Let $\phi : G \to G'$ be a homomorphism. Let $a \in G$ such that $o(a) = n$ and $o(\phi(a)) = m$. Show that $o(\phi(a)) / o(a)$. Further show that ϕ is 1-1 iff $m = n$. 5

(*b*) Show that $\frac{S_n}{A_n} \simeq W$, where $W = \{1, -1\}$ is a multiplicative group.

Hence prove the set A_n of all even permutations of S_n is a normal subgroup of S_n. 5

Section III

5. (*a*) Prove that the number of conjugate classes in S_n is $p(n)$, the number of partitions of n. 6

(*b*) Let Z denote the centre of a group G. If $\frac{G}{Z}$ is cyclic, prove G is abelian. 3

6. (*a*) Let G be a finite abelian group such that $p \mid o(G)$, where p is a prime number. Prove that $\exists\ a \in G$ such that $a \neq e$ and $a^p = e$. 6

(*b*) If p is a prime number and G is a non-abelian group of order p^3, then prove that centre of G has exactly p elements. 3

Section IV

7. (*a*) Prove that G is the internal direct product of the normal subgroups $N_1, \ldots\ldots\ldots, N_n$ iff : 4

(1) $G = N_1, \ldots\ldots\ldots, N_n$

(2) $N_i \cap (N_1 N_2 \ldots N_{i-1} N_{i+1} \ldots N_n) = (e); i = 1, \ldots, n$.

(*b*) Prove a finite abelian group is direct product of its Sylow subgroups. 5

8. (*a*) Let G be a finite group and p a prime number such that $p \mid o(G)$. Show that a Sylow p-subgroup in G is unique iff it is normal in G. 4

(*b*) Find all non-abelian groups of order 6. 5