[This question paper contains 2 printed pages.]

Sr. No. of Question Paper : 8813 **C** **Roll No............**

Unique Paper Code : 235304

Name of the Paper : III - 3 – Algebra – II

Name of the Course : B.Sc. (Hons.) Maths, Part II
(Admissions of 2011 and onwards)

Semester : III

Duration : 3 Hours

Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.

1. (a) Define a group. Prove that the set

$$G = \left\{ \begin{bmatrix} x & x \\ x & x \end{bmatrix} \middle| \; x \in R, x \neq 0 \right\}$$

is an infinite Abelian group under matrix multiplication.

(b) Let H be a non-empty finite subset of a group G. Prove that H is a subgroup of G if H is closed under the operation of G.

(c) Prove that

(i) the center of a group G is an Abelian subgroup of G.

(ii) G is Abelian if and only if G = Z(G) (2×6=12)

2. (a) Let $G = \langle a \rangle$ be a finite cyclic group of order n. Prove that a^k is a generator of G if and only if gcd (k,n) = 1. Hence show that the set of all generators of Z_n is U(n), where $n \geq 2$.

(b) Define order of an element. What is the order of

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ in}$$

(i) SL(2, R) (ii) SL(2,Z_p) (p, prime).

(c) If $G = \langle a \rangle$ is a finite cyclic group of order n, then prove that the order of any subgroup of G is a divisor of n; and, for each positive divisor *k of n*, G has exactly one subgroup of order *k* – namely, $\langle a^{n/k} \rangle$. (2×6.5=13)

3. (a) Show that if H is a subgroup of S_n $(n \geq 2)$ then either every member of H is an even permutation or exactly half of them are even.

(b) State and prove Lagrange's theorem for finite groups. Prove that every finite group of prime order is cyclic.

(c) Define the index of a subgroup. If G is a group and H is a subgroup of index 2 in G, prove that H is a normal subgroup of G. Show that the converse need not be true. (2×6=12)

4. (a) State and prove Orbit-Stabilizer Theorem.

(b) Prove that

(i) the commutator subgroup G′ of a group G is a normal subgroup of G.

(ii) If N is a normal subgroup of G, then G/N is Abelian if and only if $G' \leq N$. Hence or otherwise show that G/G′ is Abelian.

(c) Let G be a group and H a normal subgroup of G. Prove that the set $G/H = \{aH : a \in G\}$ is a group under the operation (aH) (bH) = abH. (2×6.5=13)

5. (a) Let G be a group and Z (G) be the centre of G. If G/Z (G) is cyclic then prove that G is Abelian.

(b) Find Aut (Z_{10}).

(c) Show that any infinite cyclic group is isomorphic to (Z, +). (2×6=12)

6. (a) Suppose that φ is an isomorphism from a group G onto a group G*. Prove that G is cyclic if and only if G* is cyclic. Hence show that Z, the group of integers under addition is not isomorphic to Q, the group of rationals under addition.

(b) Let G be a finite Abelian group and let p be a prime that divides order of G. Prove that G has an element of order p.

(c) Let φ be a group homomorphism from G onto G* then prove that G/Ker $\varphi \approx G^*$. (2×6.5=13)