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Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **8811**

Unique Paper Code : **235301** **C**

Name of the Paper : **III. 1 : Calculus II**

Name of the Course : **B.Sc. (Hons.)/Mathematics/Part II**

Semester : **III**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

All sections are compulsory.

Attempt any *Five* questions from each Section.

Section I

1. (*a*) Let f be the function defined by :

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4}, & \text{for } (x, y) \neq (0,0) \\ 0, & \text{for } (x, y) = (0,0) \end{cases}$$

Is f continuous at (0,0) ? Explain.

(*b*) Let $z = x^2 \sin(3x + y^3)$. Evaluate :

$$\frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y} \text{ at } \left(\frac{\pi}{3}, 0\right).$$ 3+2

2. If $F(x, y) = 0$ defines y implicitly as a differentiable function x, then show that :

$$\frac{dy}{dx} = -\frac{F_x}{F_y},$$

provided $F_y \neq 0$. Hence or otherwise obtain $\frac{dy}{dx}$ for :

$$\sin(x + y) + \cos(x - y) = y.$$ 2+3

3. Let $f(x, y)$ be a function that is differentiable at (x_0, y_0). Then show that f has a directional derivative in the direction of the unit vector $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$ given by :

$$D_u f(x_0, y_0) = f_x(x_0, y_0)\, u_1 + f_y(x_0, y_0) u_2 .$$

Hence or otherwise find the directional derivative of

$$f(x, y) = \ln(x^2 + y^3) \text{ at } P_0 \ (1, -3),$$

in the direction of $\vec{u} = 2\vec{i} - 3\vec{j}$. 3+2

4. At a certain factory, the daily output is $Q = 60K^{1/2}L^{1/3}$ units, where K denotes the capital investment (in units of \$ 1,000) and L the size of the labour force (in worker hours). The current capital investment is \$900,000 and 1,000 worker hours of labour are used each day. Estimate the change in output that will result if the capital investment is increased by \$1,000 and labour is decreased by 2 worker hours. 5

5. Find all relative extrema and saddle points of the function :

$$f(x, y) = 2x^2 + 2xy + y^2 - 2x - 2y + 5.$$ 5

6. Given that the largest and the smallest values of $f(x, y) = 1 - x^2 - y^2$ subject to the constraint $x + y = 1$ with $x \geq 0$ and $y \geq 0$ exist, use the method of Lagrange Multiplier to find these extrema. 5

Section II

7. (*a*) Write an equivalent integral with the order of integration of reversed :

$$\int_0^4 \int_{y/2}^{\sqrt{y}} f(x, y)\, dx dy .$$

(*b*) Evaluate $\iint_D \frac{dA}{y^2 + 1}$; D is the triangle bounded by $x = 2y$, $y = -x$ and $y = 2$. 2+3

8. Express the volume of the solid bounded above by the paraboloid $z = 6 - 2x^2 - 3y^2$ and below by the plane $z = 0$ as a double integral and evaluate. 2+3

9. Use polar co-ordinates to evaluate $\iint_D xy\ dA$ where D is the intersection of the circular disks $r \le 4\cos\theta$ and $r \le 4\sin\theta$. Sketch the region of integration. 4+1

10. Find the volume of the solid D bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the plane $y + z = 2$ where $z \ge 0$. 5

11. (*a*) Let $u = x + y$, $v = x - y$. Find the image of the rectangle $0 \le x \le 6, 0 \le y \le 5$ in the uv–plane. Sketch the image.

(*b*) Express the equation $z = x^2 + y^2$ in terms of spherical co-ordinates (ρ, θ, φ). 3+2

12. Find the centroid of the solid bounded by the surface

$$z = \sqrt{x^2 + y^2}$$

and the plane $z = 9$. 5

Section III

13. Find the work done by the force field

$$\vec{F} = (x^2 + y^2)\,\vec{i} + (x + y)\,\vec{j}$$

as an object moves counterclockwise along the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$ and then back to $(1, 0)$ along the x–axis. 5

14. Show that the vector field

$$\vec{F} = (20x^3z + 2y^2)\,\vec{i} + 4xy\,\vec{j} + (5x^4 + 3z^2)\,\vec{k}$$

is conservative in $\mathbf{R}^3$ and find a scalar potential function for $\vec{F}$. 5

15. State Green's Theorem for a simply connected region in $\mathbf{R}^2$. Use Green's Theorem to find the work done by the force field

$$\vec{F}(x, y) = (3y - 4x)\,\vec{i} + (4x - y)\,\vec{j}$$

when an object moves once counterclockwise around the ellipse :

$$4x^2 + y^2 = 4.$$ 2+3

16. Let S be a surface defined by $z = f(x, y)$ and R be its projection on the xy–plane. Give the formula for the surface integral of a continuous function g defined over S assuming that f, f_x and f_y are continuous functions in R.

Using the formula, evaluate the surface integral $\iint_S g\,ds$ where

$$g(x,y,z) = xz + 2x^2 - 3xy$$

and S is that portion of the plane $2x - 3y + z = 6$ that lies over the unit square

$$R : 2 \le x \le 3,\ 2 \le y \le 3$$ 1+4

17. Compute the flux integral

$$\iint_S \vec{F}.\vec{N}\,ds, \text{ where } \vec{F} = xy\,\vec{i} + z\,\vec{j} + (x+y)\,\vec{k}$$

and S is the triangular surface cut off from the plane $x + y + z = 1$ by the co-ordinate planes. Assume $\vec{N}$ is the upward unit normal. 5

18. State Stokes' theorem. Let S be the portion of the plane $x + y + z = 1$ that lies in the first octant, and let C be the boundary of S, traversed counterclockwise as viewed from above. Verify Stokes' theorem for the surface S and the vector field :

$$\vec{F} = -\frac{3}{2}y^2\,\vec{i} - 2xy\,\vec{j} + yz\,\vec{k}.$$ 5