[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 6611 **D** **Your Roll No...............**

Unique Paper Code : 235302

Name of the Course : **B.Sc. (H) Mathematics - Part II**

Name of the Paper : Numerical Methods and Programming [Paper 3.2]

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All the **six** questions are compulsory.
3. Attempt any **two** parts from each question.
4. Marks are indicated against each question.
5. Use of Scientific Calculator is allowed.

1. (a) Verify that the function $f(x) = x^6 - 3$ has a zero on the interval $(1, 2)$. Perform three iterations of the bisection method and verify that each approximation satisfies the theoretical error bound. The exact location of the zero is $p = 3^{1/6}$.

(b) Define a fixed point of a function. Find the fixed points of $g(x) = 2x(1 - x)$. Construct an algorithm to implement the fixed point iteration scheme to find a fixed point of a function.

(c) Perform three iterations of Newton's method to find an approximate value of $(18)^{1/2}$ using the equation $x^2 - 18 = 0$ and the starting approximation as 4. (12)

2. (a) Verify that the equation $\cos x - x = 0$ has a root on the interval $(0, 1)$. Perform three iterations of the method of false position to approximate the root.

(b) Define the order of convergence of an iterative method for finding an approximation to the location of a root of $f(x) = 0$. Find the order of convergence of the Newton's method.

(c) Explain geometrically the secant method to approximate a zero of a function. Construct an algorithm to implement the secant method. (13)

3. (a) Find an LU decomposition of the matrix

$$A = \begin{bmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{bmatrix}$$

and use it to solve the system $Ax = [0 \ 4 \ 1]^T$.

(b) Starting with the initial vector $X^{(0)} = 0$, perform three iterations of Jacobi method to solve the following system of equations :

$$4x_1 + x_2 + 2x_3 = 4$$
$$3x_1 + 5x_2 + x_3 = 7$$
$$x_1 + x_2 + 3x_3 = 3$$

(c) Starting with the initial vector $X^{(0)} = 0$, perform three iterations of Gauss-Seidel method to solve the following system of equations :

$$4x + 2y - z = 1$$
$$2x + 4y + z = -1$$
$$-x + y + 4z = 1$$

(13)

4. (a) Use the Lagrange interpolation to estimate $\sin(\pi/3)$ from the following data set :

x	0	$\pi/4$	$\pi/2$
$f(x) = \sin x$	$\sin 0$	$\sin\pi/4$	$\sin\pi/2$

and establish the theoretical error bound.

(b) Find the unique interpolating polynomial of degree 2 or less for the function f(x) such that f(0) = 2, f(1) = −1, f(2) = 4, using Newton's divided difference interpolation. Hence, estimate the value of f(1.5).

(c) Find the maximum value of the step size h that can be used in the tabulation of $f(x) = e^x$ on the interval [0, 1], so that the error in the linear interpolation of f(x) is less than 5×10^{-4}. (12)

5. (a) Define the forward difference operator Δ, the backward difference operator ∇, the central difference operator δ and the averaging operator μ. Prove that :

(i) $\sqrt{1 + \delta^2\mu^2} = 1 + \frac{\delta^2}{2}$

(ii) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{1}{4}\delta^2}$

(b) Derive the following backward difference approximation formula for the first derivative :

$$f'(x_0) \approx \frac{3f(x_0) - 4f(x_0 - h) + f(x_0 - 2h)}{2h}$$

where h is the spacing between the points.

(c) Verify that the second-order central difference approximation formula for the second derivative given by

$$f''(x_0) \approx \frac{f(x_0 - h) - 2f(x_0) + f(x_0 + h)}{h^2}$$

provides the exact value of the second derivative, regardless of the value of h, for the functions f(x) = x, $f(x) = x^2$ and $f(x) = x^3$, but not for the function $f(x) = x^4$. (12)

6. (a) Find an approximate value of the integral $\int_0^1 \frac{1}{1 + x^2}\,dx$ using the trapezoidal rule and obtain the theoretical error bound.

(b) Find an approximate value of the integral $\int_0^1 e^{-x}\,dx$ using the Simpson's rule. Write the degree of precision.

(c) Apply the Euler's method to approximate the solution of the initial value problem :

$$x'(t) = tx^3 - x, \qquad 0 \le t \le 1,$$

$$x(0) = 1,$$

over the interval [0, 1] using four steps. (13)