[This question paper contains 2 printed pages.]

Sr. No. of Question Paper : 6613 **D** **Your Roll No...............**

Unique Paper Code : 235304

Name of the Course : **B.Sc. (Hons.) Mathematics**

Name of the Paper : III.3 – Algebra II

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory.

1. (a) For a fixed point (a,b) in R^2, define $T_{a,b} : R^2 \to R^2$ by $T_{a,b}(x,y) = (x+a, y+b)$. Let $G = \{ T_{a,b} : a, b \in R\}$. Prove that G is a group under function composition. Is G abelian ? (6)

   (b) (i) Let G be a group and H be a non-empty subset of G. Prove that H is a subgroup of G iff a. b $\in$ H for all a, b $\in$ H and $a^{-1} \in$ H for all a $\in$ H. (3)

   (ii) Let G be a group that has exactly elements of order 3. How many subgroup of order 3 does G have ? (3)

   (c) Define centre Z(G) of a group.

   (i) Prove that Z(G) is an subgroup of G.

   (ii) Determine the centre of the group GL(2, R) (6)

2. (a) Prove that every subgroup of a cyclic group is cyclic. If G is cyclic group of order n and H is a subgroup of G, then prove that |H| divides |G|. (6)

   (b) Prove that $H = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in Z \right\}$ is a cyclic subgroup of GL(2, R). (6)

   (c) (i) Find the smallest subgroup of Z containing 6 and 15. (3)

   (ii) Let G be a group such that $x^2 = e$ for all $x \in G$, where e is the identity of G. Prove that G is abelian. (3)

3. (a) Prove that the set A_n of all even permutations of $S_n(n \geq 2)$ is a normal subgroup of S_n and $|A_n| = |S_n|/2$. (6.5)

(b) (i) Given that the orders of the elements of A_4 are 1, 2, 3. Prove that $|Z(A_4)| = 1$. (3)

(ii) State Lagrange's Theorem. Suppose $|G| = p.q$ where p and q are prime numbers. Prove every proper subgroup of G is cyclic. (3.5)

(c) (i) Let H be a subgroup of G and let $a,b \in G$. Does $aH = bH \Rightarrow a = b$? Justify your answer. (3)

(ii) If N is a normal subgroup of G and H is any subgroup of G. Prove that NH is a subgroup of G and N is normal in NH. (3.5)

4. (a) Let H be a subgroup of G. Prove that H is normal in G iff $aHbH = abH$ for all $a,b \in G$. (6.5)

(b) Let N be a normal subgroup of G and let H be a subgroup of G containing N. Prove that H/N is a normal subgroup of G/N iff H is a normal subgroup of G. (6.5)

(c) Define a commutator subgroup G′ of G. Prove that G′ is the smallest subgroup of G such that G/G′ is abelian. (6.5)

5. (a) State and prove Cayley's Theorem. (6)

(b) Let G be acyclic group of order n. Prove that *Aut* G $\approx U_n$. (6)

(c) (i) Prove that U(10) is not isomorphic to U(12). (3)

(ii) Let G be a group. Prove that the mapping $\alpha : G \to G$ defined as $\alpha(g) = g^{-1}$ for all $g \in G$ is an automorphism iff G is abelian. (3)

6. (a) Let H and K be two subgroup of G such that K is normal in G. Prove that $HK/K \approx H/H \cap K$. (6.5)

(b) Let ϕ be a homomorphism from a group G to a group $\bar{G}$ and let $g \in G$. If $\phi(g) = \bar{g}$ then prove that $\phi^{-1}(\bar{g}) = g \text{ Ker } \phi$. (6.5)

(c) Determine all homomorphisms from Z_{20} to Z_8. (6.5)