[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 2065 **GC-3** **Your Roll No...............**

Unique Paper Code : 32351302

Name of the Paper : C6 Group Theory 1

Name of the Course : **B.Sc. (Hons) Mathematics – CBCS**

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on the receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory.

1. (a) Describe the elements of the dihedral group D_3 and show that they form a group under composition of mappings. (6)

   (b) Prove that if H and K are subgroups of G, then so is $H \cap K$. Is $H \cup K$ a subgroup of G ? Justify your answer. (6)

   (c) Define Centralizer C(G) of a group G. Is C(G) Abelian ? Justify your answer. (6)

2. (a) Let $G = \langle a \rangle$ be a cyclic group of order n. Show that $G = \langle a^k \rangle$ if and only if gcd(k,n) = 1. (6)

(b) List the subgroups of Z_{30} and their generators. (6)

(c) (i) Let G be an Abelian group with identity e. Prove that

$H = \{x \in G | x^2 = e\}$ is a subgroup of G. (3)

(ii) Let a and b be elements of a group. If $|a| = 10$ and $|b| = 21$, show

that $\langle a \rangle \cap \langle b \rangle = \{e\}$. (3)

3. (a) Prove that every permutation on a finite set can be written as a cycle or product of disjoint cycles. (6)

(b) Prove that the set of even permutations in S_n forms a subgroup of S_n. (6)

(c) Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 5 & 4 & 7 & 6 & 8 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}$

Write α and β as

(i) products of disjoint cycles.

(ii) products of 2 – cycles. (6)

4. (a) Prove that any finite cyclic group of order n is isomorphic to $\mathbb{Z}_n$ and any infinite cyclic group is isomorphic to $\mathbb{Z}$. (6½)

(b) Let $\overline{G}$ denote the left regular representation of the group G (as defined in Cayley's Theorem). Calculate $\overline{U(12)}$. (6½)

(c) Give statements only, of Lagrange's Theorem and its converse. Is the converse true ? Justify your answer. (6½)

5. (a) Let H and K be subgroups of a finite group G. Prove that

$$|HK| = \frac{|H||K|}{|H \cap K|}$$ (6½)

(b) Let G be a finite Abelian group and let p be a prime number that divides the order of G. Prove that G has an element of order p. (6½)

(c) Let φ be a homomorphism from a group G to a group $\bar{G}$ and let H be a subgroup of G. Prove that

(i) if H is cyclic, then $\varphi(H)$ is cyclic. (2)

(ii) if H is Abelian, then $\varphi(H)$ is Abelian. (2)

(iii) if H is normal in G, then $\varphi(H)$ is normal in $\varphi(G)$. (2½)

6. (a) State and prove The Second Isomorphism Theorem. (6½)

(b) Let G be a subgroup of some dihedral group. For each x in G, define

$$\varphi(x) = \begin{cases} +1 & \text{if x is a rotation} \\ -1 & \text{if x is a reflection} \end{cases}$$

Prove that φ is a homomorphism from G to the multiplicative group $\{+1, -1\}$. What is the kernel of φ? (6½)

(c) Determine all homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{30}$. (6½)