[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 2064 **GC-3** **Your Roll No...............**

Unique Paper Code : 32351301

Name of the Paper : C5 Theory of Real Functions

Name of the Course : **B.Sc. (Hons.) Mathematics – C.B.C.S.**

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on the receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **three** parts from each question.

1. (a) Use the $\varepsilon - \delta$ definition of limit of a function to find $\lim\limits_{x\to 1} \dfrac{x^2 - x + 1}{x + 1}$. (5)

(b) State Sequential Criterion for Limits. Using Sequential Criterion for limits, prove that $\lim\limits_{x\to 0} \sin\dfrac{1}{x^2}$ does not exist. (5)

(c) State Squeeze Theorem. Use the theorem to show that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$. (5)

(d) Let f be defined on $A \subseteq R$ to R and let c be a cluster point of A. If $\lim\limits_{x\to c} f > 0$, then show that there exists a neighbourhood $V_\delta(c)$ of c such that $f(x) > 0$ for all $x \in A \cap V_\delta(c)$, $x \neq c$. (5)

2. (a) Let f be defined on $A \subseteq R$ to R and let c be a cluster point of $A \cap (c,\infty)$ and $A \cap (-\infty,c)$. Then show that $\lim_{x\to c} f(x) = L$ exists if and only if $\lim_{x\to c+} f(x) = L = \lim_{x\to c-} f(x)$. (5)

(b) Prove that

(i) $\lim_{x\to\infty} \frac{1}{x^2} = 0$

(ii) $\lim_{x\to 0} \frac{1}{x^2} = \infty$ (5)

(c) Let g: $R \to R$ be defined by

$$g(x) = \begin{cases} x, & \text{for } x \text{ rational} \\ 0, & \text{for } x \text{ irrational} \end{cases}$$

Find all the points at which g is continuous. (5)

(d) Determine the points of continuity of the function $f(x) = [x]$, $x \in R$, where $[x]$ denotes the greatest integer $n \in \mathbb{Z}$ such that $n \leq x$. (5)

3. (a) Let $A, B \subseteq R$. Let $f : A \to R$ and $g : B \to R$ be functions such that $f(A) \subseteq B$. If f is continuous at $c \in A$ and g is continuous at $b = f(c) \in B$ then show that the composite function $gof : A \to R$ is continuous at c. (5)

(b) Let f, g be continuous from R to R and suppose that $f(r) = g(r)$ for all rational numbers r. Show that $f(x) = g(x)$ for all $x \in R$. (5)

(c) Let f be a continuous real valued function defined on [a, b]. By assuming that f is a bounded function, show that f attains its maximum value on [a, b]. (5)

(d) Suppose that f is continuous on [0,2] and that f(0) = f(2). Prove that there exist x, y in [0,2] such that $|y - x| = 1$ and f(x) = f(y). (5)

4\. (a) Let I be a closed and bounded interval and let $f : I \to R$ be continuous on I. Then prove that f is uniformly continuous on I. (5)

(b) Show that the function $f(x) = x^2$ is not uniformly continuous on R. (5)

(c) State and prove Caratheodory's Theorem. (5)

(d) Let f: R → R be defined by

$$f(x) = \begin{cases} x^2, & \text{for } x \text{ rational} \\ 0, & \text{for } x \text{ irrational} \end{cases}$$

Show that, f is differentiate at x = 0 and find f′(0). (5)

5\. (a) Let f be continuous on [a,b] and differentiable on (a,b). Prove that if $f'(x) < 0 \; \forall x \in [a,b]$, then f is strictly decreasing on [a,b]. Is the converse true ? Justify. (5)

(b) Let g: [−1,1] → R be defined by

$$g(x) = \begin{cases} 2, & 0 < x \le 1 \\ 0, & x = 0 \\ 1, & -1 \le x < 0 \end{cases}$$

Show that g is not the derivative on [−1,1] of any function. (5)

(c) Find the Taylor series for $\sin x$ and indicate why it converges to $\sin x \ \forall x \in R$. (5)

(d) Define a convex function on $[a,b]$. Show that the function $f(x) = |x|$, $x \in [-1,1]$ is convex but not differentiable on $[-1,1]$. (5)