This question paper contains 4 printed pages]

*Your Roll No.* ....................................

9661

# B.A./B.Sc. (Hons.)/III B

## MATHEMATICS—Unit 12

## (Algebra—III)

*Time : 2 Hours* *Maximum Marks* : 38

*(Write your Roll No. on the top immediately on receipt of this question paper.)*

Attempt *one* question from each Section.

### Section I

1. (*a*) Let $Z_p$ be the ring of integers modulo $p$. Show that $Z_p$ is a field iff $p$ is a prime number. 4

(*b*) Define characteristic of an integral domain R. Show that the characteristic of R is either zero or a prime number. 5½

P.T.O.

2. (*a*) If R is a commutative ring and $a \in$ R. Then show that :

$$\langle a \rangle = \{ar + na / r \in \mathrm{R}, n \in \mathbf{Z}\},$$

where $\langle a \rangle$ denotes the smallest ideal of R containing $a$. 4

(*b*) Let R be a non-commutative ring with unity. Prove that **Z**(R), the centre of R is a subring of R. Is it an ideal ? Justify. 3½

(*c*) Give an example of a ring R with unity having a subring with unity, which is different from the unity of R. 2

**Section II**

3. (*a*) Let R be a P.I.D. (Principal Ideal Domain) which is not a field. Show that any proper ideal of R is a maximal ideal iff it is generated by an irreducible element of R. 5½

(*b*) Let C[0, 1] be the ring of all continuous real valued functions defined on [0, 1], under pointwise addition and multiplication. Let :

$$\mathrm{M} = \left\{ f \in \mathrm{C}[0,1] \middle/ f\left(\frac{2}{3}\right) = 0 \right\}.$$

Show that M is a maximal ideal of C[0, 1]. 4

4. (*a*) Prove that in a P.I.D. (Principal Ideal Domain) an element is prime iff it is irreducible. 5

(*b*) Let A and B be two ideals of a ring R. Show that :

$$\frac{A+B}{A} \approx \frac{B}{A \cap B}.$$ 4½

**Section III**

5. (*a*) Prove that every Euclidean Domain is a P.I.D. (Principal Ideal Domain). 5

(*b*) State and prove Eisenstein's criterion for irreducibility of polynomials. 4½

6. (*a*) Prove that every P.I.D. (Principal Ideal Domain) is a U.F.D. (Unique Factorization Domain). 6

(*b*) Let R be an integral domain with unity. Prove that every irreducible element of R[$x$] is also an irreducible polynomial. Give an example to show that the converse is not true. 3½

## Section IV

7. (*a*) If K is a finite extension of a field F and L is a finite extension of K, then prove that L is a finite extension of F and [L : F] = [L : K} × [K : F]. 5

(*b*) Find the degree of the splitting field of the polynomial $x^4 + 1$ over the field Q of rational numbers. 4½

8. (*a*) Prove that a regular hexagon is constructible using ruler and compass. 4

(*b*) Let L be an algebraic extension of a field K and K be an algebraic extension of F. Prove that L is an algebraic extension of F. 5½