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Your Roll No. ....

9662

B.A./B.Sc. (Hons.)/III **B**

MATHEMATICS—Unit XIII

(Algebra–IV)

Time : 2 Hours *Maximum Marks : 38*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt *two* questions from each Section.

Section I

1. Define basis of a vector space and write a basis for each of the following vector spaces. Also write the dimension of the vector space.

   (*i*) The vector space $\mathbf{C}^2$ over $\mathbf{R}$;

   (*ii*) The vector space of *n*-tuples over the field $\mathbf{C}$ of complex numbers.

   (*iii*) The vector space F[*x*] of all polynomials over a field F.

(*iv*) The vector space $M_{2\times3}(F)$ of all 2×3 matrices defined over the field F. 4½

2. (*i*) If W_1 and W_2 are two subspaces of a vector space V(F), then show that $W_1 + W_2$ is a subspace of V(F). When do we say that $W_1 + W_2$ is a direct sum ? 2½

(*ii*) Find two subspaces A and B of $\mathbf{R}^4(\mathbf{R})$ such that $\dim A = 3$, $\dim B = 2$ and $\dim (A \cap B) = 1$. 2

3. Let V be a finite dimensional vector space over F. Let W be a subspace of V. Show that :

$$\dim \frac{V}{W} = \dim V - \dim W.$$

Let V be the space of 2×2 matrices over F. Let W be the subspace of V spanned by :

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Find a basis of $\frac{V}{W}$. 4½

Section II

4. Define Rank and Nullity of a linear transformation $T : V \to W$, where V and W are finite dimensional vector spaces over a field F. Prove that :

$$\text{rank } (T) + \text{nullity } (T) = \dim V. \qquad 5$$

5. Let T be a linear operator on $\mathbf{R}^2$ defined by $T(x_1, x_2) = (-x_2, x_1)$. Let $\beta = \{\varepsilon_1 = (1, 0), \varepsilon_2 = (0, 1)\}$ and $\beta' = \{\alpha_1 = (1, 2), \alpha_2 = (1, -1)\}$ be two ordered bases for $\mathbf{R}^2$. Find a matrix P such that :

$$[T]_{\beta'} = P^{-1}[T]_{\beta}P. \qquad 5$$

6. Find range, rank, kernel and nullity of the linear transformation defined by $T : \mathbf{R}^3 \to \mathbf{R}^3$ such that

$$T(x, y, z) = (x + z, x + y + 2z, 2x + y + 3z). \qquad 5$$

Section III

7. A vector space V over F is an inner product space and W is a subspace of V. Define orthogonal complement $W^{\perp}$ of W and prove that $V = W \oplus W^{\perp}$. 4½

8. Obtain an orthonormal basis for V, the space of all real polynomials of degree at most 2, the inner product being defined by

$$< f, g > = \int_0^1 f(x)\, g(x)\, dx$$ 4½

9. (*i*) Define annihilator A(W) of a subspace W of a vector space V(F) and prove that :

$$A(W_1 + W_2) = A(W_1) \cap A(W_2).$$ 2½

(*ii*) If W = L{(1, 2, 3), (0, 4, −1)} is a subspace of $\mathbf{R}^3(\mathbf{R})$. Find A(W) and a basis of A(W). 2

Section IV

10. Find the eigenvalues, eigenvectors and the eigenspaces of the matrix :

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}.$$

Is A diagonalizable ? Justify. 5

11. (*i*) Prove that if E is the projection on R along N, then I − E is the projection on N along R. 3

(*ii*) Find a projection E which projects $\mathbf{R}^2$ onto the subspace spanned by (1, −1) along the subspace spanned by (1, 2). 2

12. If $V = W_1 \oplus W_2 \oplus \ldots\ldots \oplus W_k$, then prove that there exist k linear operators E_1, k_2, ... , E_k on V such that :

(*i*) Each E_i is a projection *i.e.* $E_i^2 = E_i$;

(*ii*) $E_i E_j = 0$ for $i \neq j$;

(*iii*) $I = E_1 + E_2 + \ldots\ldots + E_k$;

(*iv*) The range of E_i is W_i for $i = 1, 2, \ldots\ldots k$. 5