[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 6618 **D** **Your Roll No...............**

Unique Paper Code : 235503

Name of the Course : **B.Sc. (Hons.) Mathematics**

Name of the Paper : Analysis – IV (5.2)

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question.

1. (a) Show that $d : R^2 \times R^2 \rightarrow R$ defined as follows is a metric on R^2

$$d(a,a) = 0 \qquad \forall\ a \in R^2$$

For $a \neq b$, $a = (a_1, a_2)$, $b = (b_1, b_2)$

$$d(a,b) = \begin{cases} \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}, & \text{if neither a nor b is origin} \\ 1 + \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}, & \text{otherwise} \end{cases}$$

(b) Define an isometry. Prove that

(i) every isometry is injective

(ii) inverse of a surjective isometry is also an isometry

P.T.O.

(c) Suppose (X, d) is a metric space, a, b $\in$ X and S is a nonempty subset of X. Prove that

$$|\text{dist}(a, S) - \text{dist}(b, S)| \leq d(a,b) \leq \text{dist}(a, S) + \text{diam } S + \text{dist}(b, S)$$ (6+6)

2. (a) Suppose X is a metric space and A and B are subsets of X with $A \subseteq B$. Prove that

(i) $A \cap \text{iso}(B) \subseteq \text{iso}(A)$

(ii) $\text{acc}(A) \subseteq \text{acc}(B)$

(b) Suppose A is a subset of a metric space X. Prove that $\text{diam}\left(\overline{A}\right) - \text{diam}(A)$. Give an example to show that $\text{diam}(A^0) \neq \text{diam}(A)$.

(c) Suppose X is a metric space, S is a subset of X and a is an isolated point of S. Show that a is a boundary point of S if and only if $a \notin \text{iso}(X)$. Also show that

$$\partial S = \overline{S} \cap \overline{S^c}$$ (6+6)

3. (a) Suppose X is a metric space and $S \subseteq X$. Prove that S is the smallest closed superset of S.

(b) (i) Suppose X is a metric space and $S \subseteq X$. Prove that S has empty interior, if, and only if, S has dense complement.

(ii) Define an open cover for a metric space. Give an open cover for R endowed with the usual metric.

(c) Prove that the square $\{a = (a_1, a_2) \in R^2 : a_1, a_2 \in (-1,1)\}$ is an open subset of R^2 with Euclidean metric. (6+6)

4. (a) Suppose (X, d) is a metric space, $a \in X$ and $r \in R^{+}$. Prove that

$$\partial(b[a,r)) \subseteq \{x \in X : d(a,x) = r\}$$

Deduce that b[a, r) is an open set.

(b) Suppose X is a metric space, $z \in X$ and $\{x_n\}$ is a sequence in X which converges to z in X. Prove that $\cap\left\{\overline{\{x_n : n \in S\}}, S \subseteq \mathbb{N}, S \text{ infinite}\right\} = \{z\}$.

(c) Suppose X is a metric space, $z \in X$ and S is a nonempty subset of X. Prove that $z \in \overline{S}$ if and only if there is a sequence in S that converges to z in X. (6½+6½)

5. (a) Prove that every Cauchy sequence in a metric space is bounded. Is every convergent sequence also bounded ?

(b) Suppose X is a nonempty set, (Y, e) is a nonempty metric space and B(X, Y), the collection of all bounded functions from X to Y endowed with its usual supremum metric. Suppose $\{f_n\}$ is a bounded sequence in B(X, Y) such that for each $x \in X$ the sequence $\{f_n(x)\}$ converges in Y. Prove that the function $g : X \to Y$ given by

$$g(x) = \lim f_n(x)$$

for all $x \in X$, is bounded.

(c) Prove that a subset of a complete metric space is closed if and only if it is complete. (6½+6½)

6. (a) Suppose X and Y are metric spaces and $f : X \to Y$ is continuous at every point of X. Prove that for every open set V of Y, the set $f^{-1}(V)$ is open in X.

P.T.O.

(b) Suppose (X, d) and (Y, e) are metric spaces and $\{f_n\}$ is a sequence of continuous functions from X toY that converges uniformly to a function $g : X \to Y$. Prove that g is continuous.

(c) Suppose X is a metric space, S is a connected subset of X and A a subset of X such that $S \subseteq A \subseteq \overline{S}$. Prove that A is connected. Deduce that the closure of a connected set is connected. (6½+6½)