[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 6619 **D** **Your Roll No...............**

Unique Paper Code : 235504

Name of the Course : **B.Sc. (H) MATHEMATICS – III**

Name of the Paper : Algebra IV – Paper 5.3

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **five** parts form question **1.** Each part carries **three** marks.
3. Attempt any **two** parts from each of the questions **2** to **6.** Each part carries **six** marks.

1. (a) Find the splitting field of $x^4 + x^2 + 1$ over $\mathbb{Q}$.

(b) Prove that $\sin \theta$ is constructible if and only if $\cos \theta$ is constructible.

(c) Prove that if $a > 0$ is constructible number, then $\sqrt{a}$ is also constructible.

(d) Let T be the linear operator on $P_2(\mathbb{R})$ defined by $T(f(x)) = f(x) + (x + 1)f'(x)$. Find the characteristic polynomial of T.

(e) Let $V = P_3(\mathbb{R})$ be a vector space and T is defined by $T(f(x)) = f'(x) + f''(x)$ be a linear operator on V. Test T for diagonalizability.

(f) Let T be a linear operator on $\mathbb{R}^3$ defined by $T(a, b, c) = (-b + c, a + c, 3c)$. Determine the T cyclic subspace generated by $e_1 = (1,0,0)$.

(g) Let T be a linear operator on $\mathbb{R}^2$ defined by $T(a,b) = (a + 2b, -2a + b)$. Find the characteristic polynomial of T.

(h) Let T be a linear operator on a finite dimensional vector space. Prove that E_λ, the eigen space of λ, is T-invariant subspace, for any eigenvalue λ of T.

2. (a) Let F be a field and let $p(x) \in F[x]$ be irreducible over F. If a is zero of p(x) in some extension E of F, then F(a) is isomorphic to $F[x]/\langle p(x) \rangle$.

(b) Suppose that F is a field and every irreducible polynomial in F[x] is linear. Show that F is algebraically closed.

(c) Show that a finite field cannot be algebraically closed.

3. (a) If K is an algebraic extension of E and E is algebraic extension of F, then show that K is an algebraic extension of F and set of all elements of E that are algebraic over F is a subfield of E.

(b) Let E be algebraic closure of F. Show that every polynomial in F[x] splits in E.

(c) If g(x) is irreducible over GF(p) and g(x) divides $x^{p^n} - x$, prove that deg g(x) divides n.

4. (a) Find all eigenvectors of matrix $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$.

(b) Let V be an inner product space and $S = \{w_1, w_2 \ldots \ldots \ldots, w_n\}$ be a linearly independent subset of V. Define $S' = \{v_1, v_2, \ldots, v_n\}$, where $v_1 = w_1$ and

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j$$

for $2 \leq k \leq n$. Then show that S' is an orthogonal set of nonzero vectors such that span (S') = span(S).

(c) For the data $\{(-3,9), (-2,6), (0,2), (1,1)\}$, use the least squares approximation to find the best fit quadratic polynomial. Also, compute the error.

5. (a) Let V be a finite-dimensional inner product space over F, and let $g: V \to F$ be a linear transformation. Then, prove that there exists a unique vector $y \in V$ such that $g(x) = \langle x.y \rangle$ for all $x \in V$.

(b) Let $V = P(\mathbb{R})$ with the inner product

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(t)g(t)\,dt$$

and consider the subspace $P_2(\mathbb{R})$ of V with the standard ordered basis β. Use Gram-Schmidt process to replace β by an orthonormal basis $\{v_1, v_2, v_3\}$ for $P_2(\mathbb{R})$.

(c) Let T be a linear operator on a finite-dimensional vector space V, and let p(t) be a minimal polynomial of T. Show that a scalar λ is an eigenvalue of T if and only if $p(\lambda) = 0$.

6. (a) Let V be a finite-dimensional inner product space, and T be linear on V. Prove that if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

(b) Show that as a group $GF(p^n)$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p$ (n factors).

(c) Let T be a linear operator on an inner product space V. Prove that $||T(x)|| = ||x||$ for all $x \in V$ if and only if $\langle T(x), T(y)\rangle = (x,y)$ for all $x, y \in V$.