[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 1119 **G** **Your Roll No.....**

Unique Paper Code : 235504

Name of the Paper : Algebra IV (MAHT-503)

Name of the Course : **B.Sc. (H) MATHEMATICS – III**

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **five** parts from Question **1.** Each part carries **three** marks.
3. Attempt any **two** parts from each of the Questions **2** to **6.** Each part carries **six** marks.

1. (i) Show that $\mathbb{Q}\left(\sqrt{3},\sqrt{5}\right)=\mathbb{Q}\left(\sqrt{3}+\sqrt{5}\right)$.

(ii) Find a basis for $\mathbb{Q}\left(\sqrt{2i}\right)$ over $\mathbb{Q}$.

(iii) Prove that an angle θ is constructible if and only if $\sin\theta$ is constructible.

(iv) Find the dual basis β^* of an ordered basis $\beta = \{(2, -1), (1, -1)\}$ for $\mathbb{R}^2$.

(v) Find all the eigen vectors of the matrix $A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$.

(vi) Let β be a basis for a finite-dimensional inner product space. Prove that if $\langle x,z\rangle = \langle y,z\rangle$ for all $z \in \beta$, then $x = y$.

P.T.O.

(vii) Let $S = \{(1, i, 0), (1,1,2)\}$ in $\mathbb{C}^3$. Compute $S^{\perp}$.

(viii) For inner product space V and linear operator T on V, evaluate T^* at a vector x in V where $V = \mathbb{R}^2$, $T(a, b) = (a + 2b, -3a + b)$, $x = (5,3)$.

2. (a) Let F be a field and f(x) be a non-constant polynomial in F[x]. Show that there is an extension field E of F in which f(x) has a zero.

(b) Prove or disprove that $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{-5})$ are field-isomorphic.

(c) Suppose that E, is an extension of F of prime degree. Show that, for every a in E, F(a) = F or F(a) = E.

3. (a) Prove that $\cos 2\theta$ is constructible if and only if $\sin \theta$ is constructible.

(b) Let a and b belong to some extension of F and let b be algebraic over F. Prove that $[F(a,b) : F(a)] \leq [F(a,b): F]$.

(c) Show that no finite field is algebraically closed.

4. (a) Let W_1 and W_2 be two subspaces of a vector space. Prove that

$$(W_1 + W_2)^0 = W_1^0 \cap W_2^0$$

(b) Let T be the linear operator on $\mathbb{R}^4$ defined by

$$T(a,b,c,d\} = (a + b + 2c - d,\ b + d,\ 2c - d,\ c + d)$$

and let $W = \{(t, s, 0,0) : t, s \in \mathbb{R}\}$.

Show that the characteristic polynomial of T_W, the restriction of T to W, divides the characteristic polynomial of T.

(c) Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 & + 4a_3 \\ 2a_1 - 3a_2 & + 2a_3 \\ 4a_1 & + a_3 \end{pmatrix}$$

Find all the eigen vectors of T.

5. (a) Let $\{v_1, v_2, ..., v_k\}$ be an orthonormal set in an inner product space V, and let $a_1, a_2, ..., a_k$ be scalars. Prove that

$$\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \sum_{i=1}^{k} |a_i|^2$$

(b) Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Then prove that there exists a unique function $T^*: V \to V$ such that

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle \text{ for all } x, y \in V.$$

Further, prove that T^* is linear.

(c) For inner product space V and linear operator T on V, evaluate T^* at a vector f in V where,

$$V = P_1(\mathbb{R}) \text{ with } \langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\, dt, \quad T(f) = f' + 3f, \quad f(t) = 4 - 2t$$

6. (a) Let T be a linear operator on a finite-dimensional vector space, and let p(t) be the minimal polynomial of T. Prove that T is not invertible if and only if $p(0) = 0$.

(b) Let T be a linear operator on an inner product space V. If $\langle T(x), y\rangle = 0$ for all $x, y \in V$, prove that $T = T_0$ (the zero operator on V). Prove that the same result is true if the equality holds for all x and y in some basis of V.

(c) Let K be a finite extension field of a finite field F. Show that there is an element a in K such that = F(a).