Sl. No. of Ques. Paper : **1357**   **F–7**

Unique Paper Code : **2351504**

Name of Paper : **Probability & Statistics**

Name of Course : **B.Sc. (Hons.) Mathematics (Erstwhile FYUP)**

Semester : **V**

Duration : : **3 hours**

Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

In all there are six questions. Question No. 1 is compulsory and it contains seven parts of 3 marks each, out of which any five parts are to be attempted. In Question Nos. 2 to 6 , attempt any two parts from three parts. Each part carries 6 marks.

Use of scientific calculator is allowed.

Q. 1 Attempt any five parts.

(i) There are 5 red chips and 3 blue chips in a bowl. The red chips are numbered 1, 2 ,3 4, 5, respectively, and the blue chips are numbered 1,2,3, respectively. If 2 chips are to be drawn at random and without replacement, find the probability that these chips have either the same number or the same color.

(ii) Verify that $b(x; n, \theta) = b(n-x; n, 1 - \theta)$

(iii) Suppose X has the pdf

$$f_X(x) = \begin{cases} cx^3 & 0 < x < 2 \\ 0 & elsewhere \end{cases}$$

then find $P\left(\dfrac{1}{4} < X < 1\right)$.

(iv) Let X_1 and X_2 have the joint pmf $p(x_1, x_2)$ described as follows:

(x_1, x_2)	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)
$p(x_1, x_2)$	1/18	3/18	4/18	3/18	6/18	1/18

and $p(x_1, x_2)$ is equal to zero elsewhere. Find $E(X_2 \mid x_1)$. when $x_1 = 1$.

(v) Let X be a random variable such that $E[(X-b)^2]$ exists for all real b. Show that $E[(X-b)^2]$ is a minimum when $b = E(X)$.

(vi) Find the median of the distribution

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$

(vii) Suppose that the number of items produced in a factory during a week is a random variable X with mean 500. If the variance of a week's production is known to equal 100, then find the lower bound for $P\{|X - 500| < 100\}$.

Q.2 (i) Show that in the limit when $n \to \infty$, $\theta \to 0$ and $n\theta = \lambda$ remains constant, the number of successes in the binomial distribution $B(n,\theta)$ is a random variable having a Poisson distribution with the parameter λ.

(ii) Show that if X is a random variable having a binomial distribution with the parameters n and θ, then the moment generating function of

$$Z = \frac{X - n\theta}{\sqrt{n\theta(1 - \theta)}}$$

approaches that of the standard normal distribution when $n \to \infty$.

(iii) Show that the moment-generating function of the geometric distribution is given by

$$M_x(t) = \frac{\theta e^t}{1 - e^t(1 - \theta)}$$

Use it to show that $\mu = \frac{1}{\theta}$ and $\sigma^2 = \frac{1-\theta}{\theta^2}$.

Q.3 (i) Show that the rth moment about the origin of the gamma distribution is given by

$$\mu'_r = \frac{\beta^r \Gamma(\alpha + r)}{\Gamma(\alpha)}$$

and hence show that the mean and variance are given by $\mu = \alpha\beta$ and $\sigma^2 = \alpha\beta^2$.

(ii) Let X_1 and X_2 have the pdf

$$f(x_1, x_2) = \begin{cases} 8x_1 x_2 & 0 < x_1 < x_2 < 1 \\ 0 & elsewhere \end{cases}$$

Evaluate $E(7X_1 X_2^2 + 5X_2)$.

(iii) Let $f_{1,2}(x_1/x_2) = c_1 x_1/x_2^2$, $0 < x_1 < x_2$, $0 < x_2 < 1$, zero elsewhere, and

$f_2(x_2) = c_2 x_2^4$, $0 < x_2 < 1$, zero elsewhere, denote respectively, the conditional pdf of X_1

, given $X_2 = x_2$, and the marginal pdf of X_2. Determine:

(a) The constants c_1 and c_2.

(b) The joint pdf of X_1 and X_2.

(c) $P\left(\frac{1}{4} < X_1 < \frac{1}{2} \middle/ X_2 = \frac{5}{8}\right)$.

(d) $P\left(\frac{1}{4} < X_1 < \frac{1}{2}\right)$.

Q. 4 (i) Let the random variables X and Y have the joint pdf

$$f(x,y) = \begin{cases} x+y & 0 < x < 1, 0 < y < 1 \\ 0 & elsewhere \end{cases}$$

Compute   a) $P\left(X_1 \le \frac{1}{2}\right)$

b) $P(X_1 + X_2 \le 1)$

c) ρ_{xy}

(ii) Let $f(x_1, x_2) = 21 x_1^2 x_2^3$, $0 < x_1 < x_2 < 1$, zero elsewhere, be the joint pdf of X_1 and

X_2.

(a) Find the conditional mean and variance of X_1, given $X_2 = x_2$, $0 < x_2 < 1$.

(b) Find the distribution of $Y = E(X_1/X_2)$

(iii) Let the continuous type random variables X and Y have the joint pdf

$$f(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & elsewhere \end{cases}$$

Find

(a) mgf of joint distribution.

(b) $Cov(X,Y)$

(c) Are the random variables dependent?

Q.5 (i) If X and Y have a bivariate normal distribution, show that the conditional density of

Y given $X = x$ is a normal distribution with the mean

$$\mu_{Y\cdot x} = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1)$$

and the variance

$$\sigma^2_{Y\cdot x} = \sigma^2_2(1 - \rho^2)$$

(ii) Given the joint density

$$f(x,y) = \begin{cases} xe^{-x(1+y)} & x > 0 \text{ and } y > 0 \\ 0 & elsewhere \end{cases}$$

find $\mu_{Y\cdot x}$.

(iii) Given the joint density

$$f(x,y) = \begin{cases} 24xy & x > 0, y > 0 \text{ and } x + y < 1 \\ 0 & elsewhere \end{cases}$$

Find the regression equation of Y on X using the formula

$$\mu_{Y/x} = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1)$$

Q.6 (i) State and prove Chebyshev's Inequality.

(ii) Let X be the number of times that a fair coin, flipped 40 times, lands heads. Find the probability that $X = 20$. Use the normal approximation and then compare it to the exact solution.

(iii) Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with probability α ; and if it does not rain today, then it will rain tomorrow with probability β. If $\alpha = 0.7$ and $\beta = 0.4$, then calculate the probability that it will rain four days from today given that it is raining today.