[This question paper contains 4 printed pages.]

1214 *Your Roll No.* ....

B.Sc. (Hons.)/I **A**

PHYSICS – Paper IV

(Mathematics – I)

Time : 3 Hours *Maximum Marks : 38*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any **two** *parts from each question.*

1. (a) State the Cauchy's General Principle of convergence of a sequence and use it to discuss the convergence of the sequence $<a_n>$ where

$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + --- + \frac{1}{n!}$$ (3)

(b) (i) Evaluate $\lim_{n\to\infty} \frac{1}{n}\left[1 + 2^{1/2} + 3^{1/3} + --- + n^{1/n}\right]$

(ii) Prove that if $p > 0$, then

$$\lim_{n\to\infty} \frac{n^k}{(1+p)^n} = 0$$ (3)

(c) Let $<a_n>$ be a sequence defined as

$$a_1 = 1,\ a_{n+1} = \left(\frac{3 + a_n^2}{2}\right)^{1/2},\ \forall\ n \geq 1.$$

Show that the sequence $<a_n>$ converges to $\sqrt{3}$.

(3)

P.T.O.

2. (a) Discuss the convergence of two of the following series

(i) $\sum_{n=1}^{\infty} \cos\frac{1}{n}$

(ii) $\sum_{n=1}^{\infty} \frac{n+1}{n^p}$

(iii) $\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot ----- \cdot 2n}{1 \cdot 3 \cdot 5 ---- \cdot (2n+1)}$ (4)

(b) Use the Cauchy's Integral Test to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. (4)

(c) Define absolute convergence of a series and prove that every absolutely convergent series is convergent but not conversely. (4)

3. (a) (i) Prove that $\lim_{x \to 0} \frac{x e^{1/x}}{1 + e^{1/x}} = 0$.

(ii) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous at $x = 0$ (2+2=4)

(b) Prove that $f(x) = x^2$ is not uniformly continuous on $[0, \infty[$. (4)

(c) Using Taylor's Theorem, show that for $x > 0$,

$$x - \frac{x^3}{6} < \sin x < x. \quad (4)$$

4. (a) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$$

is discontinuous at (0, 0). (4)

(b) Show that for the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{otherwise} \end{cases}$$

$f_{xy}(0, 0) \neq f_{yx}(0, 0)$. (4)

(c) Show that the function $f(x, y) = y^2 + x^2y + x^4$ has (0, 0) as the only critical point and that f(x, y) has a minimum at that point. (4)

5. (a) If f and g are bounded and integrable on [a, b], then prove that their product fg is also bounded and integrable on [a, b]. (4)

(b) Show that the function $f(x) = [x]$ where $[x]$ denotes the greatest integer $\leq x$, is integrable on $[0, 3]$. Also find the value of $\int_0^3 [x]\, dx$. (4)

(c) Define a Riemann Integrable function on a closed and bounded interval $[a, b]$. Using the definition find $\int_0^1 x^2\, dx$. (4)