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Your Roll No.................

5718

B.Sc. (Hons.) PHYSICS/III Sem. B

Paper—PHHT-310

(Mathematics—I)

(Admission of 2010 and onwards)

Time : 3 Hours *Maximum Marks* : 75

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *two* questions from each Section.

Section I

1. (*a*) Define a convergent sequence. Prove that every convergent sequence is bounded. Is the converse true ?

(*b*) Show that the sequence $<r^n>$ converges to zero if $|r| < 1$. 4+3½

2. (*a*) Let $<a_n>$ be a sequence defined as follows :

$$a_1 = 1,\ a_{n+1} = \frac{4 + 3a_n}{3 + 2a_n},\ n \geq 1$$

Show that $<a_n>$ converges. What is the the limit of $<a_n>$?

(*b*) Prove that every monotonically increasing sequence that is not bounded above diverges to $+\infty$. 5+2½

3. (*a*) Show that every convergent sequence is Cauchy. Apply it to pove that the sequence $<a_n>$ defined by :

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots\dots\dots + \frac{1}{n}$$

does not converge.

(*b*) Prove that if the sequence $<a_n>$ converges to a, then the sequence $<|a_n|>$ converges to $|a|$. 5+2½

Section II

1. Let Σu_n and Σv_n be two series of positive terms such that :

$$\lim_{n\to\infty} \frac{u_n}{v_n} = l,$$

where l is finite and non-zero, then show that both the series converge or diverge together. Use the above result to test for convergence the series $\sum_{n=1}^{\infty} \sin\frac{1}{n}$. 5+2½

2. Test the convergence of the following series : 4+3½

(i) $\frac{x^2}{2\sqrt{1}} + \frac{x^3}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \ldots\ldots\ldots\ldots (x > 0)$

(ii) $\sum_{n=1}^{\infty} u_n$, where $u_n = \frac{(-1)^{n-1}}{n}\left(\sqrt{n+1} - \sqrt{n-1}\right)$. 4+3½

3. (a) Use $\in$-δ definition of limit to show that :

$$\lim_{x\to 1} \frac{x^2 - x + 1}{x + 1} = \frac{1}{2}.$$

(b) Show that $\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right)$ does not exist. 4½+3

Section III

1. Define $g : \mathbf{R} \to \mathbf{R}$ by

   $g(x) = 2x$, when x is rational

   and $g(x) = x + 3$, when x is irrational.

   Show that g is continuous at $x = 3$ and discontinuous everywhere else. 3+4½

2. Show that the function $f(x) = \frac{1}{x^2}$ is uniformly continuous on $A = [1, \infty)$ but is not uniformly continuous on $B = (0, \infty)$. 3½+4

3. (*a*) Obtain Maclaurin series expansion of :

   (*i*) $f(x) = \sin x,\ x \in \mathbf{R}$

   (*ii*) $f(x) = (1 + x)^m,\ x \in \mathbf{R}$ and $m \in \mathbf{N}$.

   (*b*) Use Mean value theorem to prove :

   $|\sin x - \sin y| \leq |x - y|$ for all x, y in $\mathbf{R}$. 5+2½ (*b*) Show that the function :

$f : [1, 2] \to \mathbf{R}$ defined by

$f(x) = 6x + 5, \; x \in [1, 2]$

is Riemann integrable on [1, 2] and find the value of

$\int_{1}^{2} f(x)\, dx.$ 3+4½

Section IV

1. State Schwarz's and Young's theorems. Show that for the function :

$$f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$f_{xy}(0, 0) = f_{yx}(0, 0)$,

but the conditions of Schwarz's and Young's theorems are not satisfied. 2+5½

2. Show that the function f, where

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & x^2 + y^2 \neq 0 \\ 0, & x = y = 0 \end{cases}$$

is continuous, possesses partial derivatives but is not differentiable at the origin. 3+1½+3

3. (*a*) Expand $x^2y + 3y - 2$ in powers of $x - 1$ and $y + 2$.

(*b*) Find all the maxima and minima of the function given by : 4+3½

$$f(x, y) = x^3 + y^3 - 63(x + y) + 12xy.$$

Section V

1. Prove that a bounded function f is Riemann integrable in the interval $[a, b]$ iff for every $\in > 0$, there exists a partition P such that : 7½

$$U(P, f) - L(P, f) < \in.$$

2. Compute $\int_{-1}^{1} f(x)\, dx$ where $f(x) = |x|$. 7½

3. (*a*) Prove that if P is a partition of $[a, b]$ and f is a bounded function on $[a, b]$, then :

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a)$$

where

$$m = \inf\{f(x) : a \leq x \leq b\} \text{ and}$$

$$M = \sup\{f(x) : a \leq x \leq b\}.$$