[This question paper contains 4 printed pages.]

1422 *Your Roll No.* ....

B.Sc. (Hons.) / II **A**

STATISTICS – Paper XII

(Probability Theory – II)

(For Admissions of 1999 and onwards)

Time : 2 Hours *Maximum Marks : 38*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt **Four** *questions in all, selecting at least* **one** *question from each Section.*

SECTION I

1. (a) If X is a random variable and $E(X^2) < \infty$, then prove that

$$P(|X| \geq a) \leq E(X^2)/a^2, \quad \forall \ a > 0.$$

Use Chebychev's inequality to show that for $n > 36$, the probability that, in n throws of a fair die, the number of sixes lies between $\frac{1}{6}n - \sqrt{n}$ and $\frac{1}{6}n + \sqrt{n}$ is atleast 31/36.

(b) If $\{X_n\}$ is a sequence of independent Bernoulli variables such that

P.T.O.

$$P(X_i = 1) = p,\ P(X_i = 0) = 1 - p,\ 0 < p < 1,$$
$$i = 1, ---, n$$

and $S_n = X_1 + X_2 + --- + X_n$

Find the distribution of S_n for large values of n. (4½,5)

2. (a) Define convergence in probability and convergence in distribution.

Prove that $X_n \xrightarrow{p} C$ iff $F_n(x) \to 0$ or 1 according as $x < C$ or $x > C$, where $F_n(x)$ is the distribution function of X_n.

(b) Let $\{X_n\}$ be a sequence of mutually independent random variables such that

$X_n = \pm 1$ with probability $\frac{1 - 2^{-n}}{2}$ and

$X_n = \pm 2^{-n}$ with probability 2^{-n-1}.

Examine whether the weak law of large numbers can be applied to the sequence $\{X_n\}$. (4½,5)

SECTION II

3. (a) What is a compound distribution ? If X has Poisson distribution

$$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!},\ r = 0, 1, ---$$

where the parameter λ is a random variable of the continuous type with the density function

$$f(\lambda) = \frac{a^{\nu} e^{-a\lambda} \lambda^{\nu-1}}{\overline{|\nu}}, \ \lambda \geq 0, \ \nu > 0, \ a > 0,$$

derive the distribution of X.

Show that the characteristic function of X is given by

$$\varphi_X(t) = E(e^{itX}) = q^{\nu}(1 - pe^{it})^{-\nu},$$

where $p = \frac{1}{1+a}$, $q = 1 - p$.

(b) Let $X_1, X_2, ---, X_{2m+1}$ be an odd-size random sample from a $N(\mu, \sigma^2)$ population. Find p.d.f. of the sample median and show that it is symmetric about μ, and hence has the mean μ. (5½,4)

4. (a) (i) Find the p.d.f. of $X_{(r)}$ in a random sample of size n from the exponential distribution :

$$f(x) = \alpha e^{-\alpha x}, \ \alpha > 0, \ x \geq 0$$

(ii) Show that $X_{(r)}$ and $W_{rs} = X_{(s)} - X_{(r)}$, $r < s$, are independently distributed.

(iii) What is the distribution of $X_{(r+1)} - X_{(r)}$?

(b) Discuss how the p.d.f. of a random variable can be obtained from its characteristic function.

Let X be a continuous random variable with its characteristic function given by

$$\varphi_X(t) = e^{-\frac{1}{2}t^2}.$$

Obtain the p.d.f. of X. (4½,5)

SECTION III

5. (a) If $(X, Y) \sim N(0, 0, 1, 1, \rho)$ then prove that

$$\rho(X > 0,\ Y > 0) = \frac{1}{4} + \frac{\sin^{-1}\rho}{2\pi}.$$

(b) Show that (X, Y) possesses a bivariate normal distribution iff every linear combination of X and Y viz., $aX + bY$, $a \neq 0$, $b \neq 0$, is a normal variate. (4½,5)

6. (a) If $X_1, X_2, ---, X_K$ are K independent Poisson variates with parameters $\lambda_1, \lambda_2, ---, \lambda_K$ respectively prove that the conditional distribution

$\rho(X_1 \cap X_2 \cap --- \cap X_K | X)$, where

$X = X_1 + X_2 + --- + X_K$ is fixed, is multinomial.

(b) If $\underset{\sim}{X} \sim N_p\left(\underset{\sim}{\mu}, \Sigma\right)$ and is partitioned as

$$\underset{\sim}{X} = \begin{pmatrix} \underset{\sim}{X}^{(1)}_{K \times 1} \\ \underset{\sim}{X}^{(2)}_{(p-K) \times 1} \end{pmatrix}$$

Derive the marginal p.d.f. of $\underset{\sim}{X}^{(1)}_{K \times 1}$. (4½,5)