This question paper contains 3 printed pages. **Your Roll No. ....**

Sl. No. of Ques. Paper : **1473** **F-7**

Unique Paper Code : **2372303**

Name of Paper : **Probability and Statistical Methods III**

Name of Course : **Erstwhile 4 year UG Programme (FYUP) Statistics**

Semester : **III**

Duration : **3 hours**

Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt 15 questions in all selecting 5 questions from each Section.
All questions carry equal marks.

SECTION A

1. Two random variables X and Y have the joint density

$$f(x, y)=\begin{cases}6x^2 y, & 0<x<1, 0<y<1\\ 0, & \text{elsewhere}\end{cases}$$

Find: (a) $P[X+Y<1]$
(b) $P[X>Y]$
(c) $P[X<1 \mid Y<2]$

2. Two unbiased dice are thrown. Let X_1 be the score on the first die and X_2 the score on the second die. Let $Y = \max(X_1, X_2)$.
(i) Write down the joint distribution of Y and X_1.
(ii) Find the mean and variance of Y and Covariance (Y, X_1).

3. Let X and Y are jointly distributed with p.d.f.

$$f_{XY}(x, y)=\begin{cases}8xy, & 0<x<y<1\\ 0, & \text{otherwise}\end{cases}$$

Find (a) $E(Y \mid X=x)$
(b) $E(XY \mid X=x)$
(c) $\text{Var}(Y \mid X=x)$.

4. Define multinominal distribution. If $X_1, X_2, X_3, \ldots\ldots X_K$ are K independent Poisson variates with parameters $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots \lambda_K$ respectively, prove that the conditional distribution $P(X_1 \cap X_2 \cap \ldots\ldots \cap X_K \mid X)$ where $X = X_1 + X_2 + \ldots + X_K$ is fixed, is multinomial.

5. State inversion theorem. Find the density function *f(x)* corresponding to the characteristic function:

$$\phi_X(t)=\begin{cases}1-|t|, & |t|\le 1\\ 0, & |t|>1\end{cases}$$

6. Prove that $F_X(x)+F_Y(y)-1 \le F_{XY}(x,y) \le \sqrt{F_X(x)F_Y(y)}$ for all x, y where symbols have their usual meanings.

SECTION B

7. Given two random variables X and Y. If U = aX + bY and V = bX – aY, show that U and V are uncorrelated if $\frac{ab}{a^2-b^2}=\frac{\rho\sigma_x\sigma_y}{\sigma_x^2-\sigma_y^2}$.

8. Prove that Spearman's rank correlation coefficient is given by $1-\frac{6\sum d_i^2}{(n^3-n)}$, where d_i denotes the difference between the ranks of i^{th} individual.

9. Given that Y = aX + b and X = cY + d are the two lines of regression of Y on X and X on Y respectively. Find:

   (a) Means of X and Y.

   (b) Correlation coefficient between X and Y.

   (c) Ratio of standard deviation of X and Y.

10. Show that the equation of the plane of regression of X_1 on X_2 and X_3 is given by $\frac{X_1}{\sigma_1}\omega_{11}+\frac{X_2}{\sigma_2}\omega_{12}+\frac{X_3}{\sigma_3}\omega_{13}=0$, where symbols have their usual meaning.

11. Explain the concept of multiple correlation. Show that the multiple correlation coefficient $R_{1.23}$ is given by:

$$R_{1.23}^2 = 1-\frac{\omega}{\omega_{11}}$$

where symbols have their usual meaning.

12. For three variables X, Y and Z, prove that

$$r_{XY}+r_{YZ}+r_{ZX} \ge -\frac{3}{2}$$

SECTION C

13. State and prove Chebyshev's inequality. Use Chebyshev's inequality to determine how many times a fair coin must be tossed in order that the probability will be at least 0·9 that the ratio of the observed number of heads to the number of tosses will lie between 0·4 and 0·6

14. Let X_n be a sequence of mutually independent variates such that:

$$P\{X_n = \pm 1\} = \frac{1}{2}\left(1 - 2^{-n}\right),\ P\{X_n = \pm 2^{-n}\} = 2^{-n-1}$$

Does the WLLN hold for this sequence?

15. Define convergence in probability and convergence with probability one. Is there any relation between them? If so, prove.

16. Define bivariate normal distribution for bivariate random variable (X, Y). Find marginal density of X. Also find conditional distribution of X given $Y = y$.

17. If (X, Y) is $BVN\left(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho\right)$, find the probability of simultaneous realization of the inequality $X > E(X), Y > E(Y)$.

18. X and Y are independent random variables, each exponentially distributed with the same parameter θ, find p.d.f. for $\frac{X}{X+Y}$ and identify its distribution.