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Your Roll No. ....

9653

B.A./B.Sc. (Hons.)/I **B**

MATHEMATICS—Unit III

(Analysis–I)

(Admissions of 2008 and before)

Time : 2 Hours *Maximum Marks : 38*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *two* parts from each question.

Each question has three parts (*a*), (*b*) and (*c*).

Section I

1. (*a*) (*i*) Define supremum of a bounded set of real numbers. Show that the supremum of a set $S \subseteq R$, whenever it exists, is unique.

(*ii*) Prove that the intersection of two neighbourhoods of a point of R is again a neighbourhood of the same point. 2,3

(*b*) Prove that the union of an arbitrary family of open sets is open. Is the result true for the intersection of an arbitrary family of open sets ? Justify your answer. 5

(c) Define a closed set of real numbers. Prove that a subset S of R is closed if and only if it contains all its limit points. 5

Section II

2. (a) (i) Let $<a_n>$ be a sequence such that :

$$a_n \neq 0 \text{ for any } n \text{ and } \frac{a_{n+1}}{a_n} \to l.$$

Prove that if $|l| < 1$ then $a_n \to 0$ as $n \to \infty$.

(ii) If

$$\lim_{n \to \infty} a_n = a$$

and $a \neq 0$, show that there exists a positive number k and a positive integer m such that $|a_n| > k$, whenever $n \leq m$. 3,2

(b) State and prove Monotone Convergence Theorem for sequences. Use it show that the sequence $<a_n>$ defined by :

$$a_1 = 1, a_n = 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots\ldots + \frac{1}{\lfloor n-1} \quad \forall\, n \geq 2$$

converges. 5

(c) Prove that if a sequence <a_n> of positive numbers converges to a limit l, then :

$$\lim_{n \to \infty} (a_1 a_2 \ldots\ldots a_n)^{\frac{1}{n}} = l$$

Deduce that if :

$$b_n = \left(\frac{2}{1} \cdot \left(\frac{3}{2}\right)^2 \cdot \left(\frac{4}{3}\right)^3 \ldots\ldots \left(\frac{n+1}{n}\right)^n \right)^{\frac{1}{n}}$$

then the sequence <b_n> converges to e. 5

Section III

3. (a) Prove that a necessary condition for convergence of a series $\sum_{n=1}^{\infty} u_n$ is that $u_n \to 0$ as $n \to \infty$.

Hence or otherwise prove that the series :

$$\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \ldots\ldots + \sqrt{\frac{n}{2(n+1)}} + \ldots\ldots$$

does not converge. 4

(b) State and prove Cauchy's nth root test for the convergence of a positive term series. 4

(*c*) Define absolute convergence of a series. Show that an absolutely convergent series is convergent. Give an example to show the converse is not always true. 4

Section IV

4. (*a*) If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, then prove that :

$$(x^2 - 1) y_{n+2} + (2n+1) x y_{n+1} + (n^2 - m^2) y_n = 0.$$ 5

(*b*) (*i*) If

$$z = \tan^{-1} \frac{x^3 + y^3}{x - y}, \ x \neq y,$$

then show that :

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \sin 2z.$$

(*ii*) Find the multiple points of the curve :

$$x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0.$$ 2,3

(*c*) Trace the curve :

$$(x^2 + y^2)x - a(x^2 - y^2) = 0 \qquad (a > 0).$$ 5