[This question paper contains 4 printed pages.]

1468-A *Your Roll No.* ....

B.A./B.Sc. (Hons.)/II **A**

MATHEMATICS – Unit IX

(Analysis – III)

(Admissions of 2008 and before)

Time : 2 Hours *Maximum Marks : 38*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any **four** *parts from Q. No.* **1** *and any* **two** *parts from Q. No.* **2** *to Q. No.* **5.**

1. (a) Show that the Identity function

   $I : (\mathbb{R}, d_u) \to (\mathbb{R}, d_s)$ is not a homeomorphism on $\mathbb{R}$, where

   d_u – usual modulus metric on $\mathbb{R}$

   d_s – discrete metric on $\mathbb{R}$.

   (b) Examine the compactness of Real Line metric space with modulus metric.

   (c) Examine the existence of the unique solution in some nbd of the pt. (–1, 1) for the equation

$$y^2 - x^3y + 2x^5 = 0.$$

   If it exists, obtain it by actual calculations.

P.T.O.

(d) Use definition of extreme value to examine the function $f(x, y) = (x - y)^2 + (x^3 - y^3) + x^5$ for extreme value at (0, 0).

(e) Let (X, d) be a disconnected metric space and $Y = \{y_1, y_2\}$ is a 2-point discrete metric space. Show the existence of a continuous function

$f : (X, d) \xrightarrow{\text{onto}} (Y, \text{Discrete})$. (2½×4=10)

2. (a) Let $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, define function

$d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ as

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

$\forall\ (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$

Show that d is a metric function on $\mathbb{R}^2$.

(b) State a criterion for a function $f : (X, d_1) \to (Y, d_2)$ to be continuous on X.

Show that any function

f : Discrete Metric Space (X, d_1) to a metric space (Y, d_2) is continuous on X.

(c) Define interior of a set A in a metric space (X, d) and show that interior of a set A is the largest open set contained in A. (3½×2=7)

3. (a) If a metric space (X, d) is complete and $\langle F_n \rangle_{n=1}^{\infty}$ is a decreasing sequence of non empty closed subsets of X with diameter (F_n) tending to zero as $n \to \infty$

Then $\bigcap_{n=1}^{\infty} F_n \neq \phi$.

(b) (i) Define a contraction mapping on a metric space (X, d) and show that it is uniformly continuous on X.

(ii) Check whether the function

$f : (\mathbb{R}, 11) \to (\mathbb{R}, 11)$ defined as

$f(x) = x^3$ is a contraction mapping or not on $(\mathbb{R}, 11)$, where $(\mathbb{R}, 11)$ is a real Line metric space with modulus metric.

(c) Define an onto isometry from metric space (X, d_1) to (Y, d_2) and show that isometric image of a complete metric space is complete. (3½×2=7)

4. (a) Show that a Compact Subset of a metric space is a closed set. Justify the statement "a closed subset of a metric space may not be compact".

(b) Define a totally bounded metric space. Show that a totally bounded metric space is bounded but a bounded metric space may not be totally bounded.

(c) (i) Define a disconnected metric space and show that a pair of non empty closed sets are separated iff they are disjoint.

P.T.O.

(ii) State one of the criteria for a metric space (X, d) to be disconnected. (3½×2=7)

5. (a) (i) f(x, y) is a real valued function defined in a domain $D \subseteq \mathbb{R}^2$, containing the point (a, b). State condition under which f(x, y) is discontinuous at (a, b).

(ii) Examine the function

$$f(x, y) = \frac{x^3 y}{x^6 + y^2} \quad (x, y) \neq (0, 0)$$
$$= 0 \quad \text{at } (0, 0)$$

for continuity at (0, 0).

(b) (i) Assuming the existence of $f_{yx}(a,b)$,

show that $f_{yx}(a,b) = \mu_{K\to 0}\ \mu_{h\to 0} \dfrac{op(h,K)}{hK}$

for some fn. op(h, K).

(ii) Examine the Equality "$f_{xy}(0,0) = f_{yx}(0,0)$"

for $f(x,y) = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y) \quad x \neq 0, \ y \neq 0$

$= 0$ elsewhere

(c) State Taylor's theorem for a real valued function f(x, y) about a point (a, b), with remainder after 2 terms. Use it to obtain Taylor's expansion for $f(x, y) = e^{ax} \sin by$ upto 2nd degree terms in x and y. (3½×2=7)