[This question paper contains 4 printed pages.]

1464-A *Your Roll No.* ....

B.A./B.Sc. (Hons.)/II **A**

MATHEMATICS – Unit V

(Algebra – II)

(Admissions of 2008 and before)

Time : 2 Hours *Maximum Marks : 38*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any **one** *question from each Section.*

SECTION I

1. (a) Prove that a semigroup G is a group if for all $a, b \in G$, the equations $ax = b$ and $xa = b$ are solvable in G.

   Give an example to show that the conclusion does not hold if only one of the equations is solvable. (5)

   (b) Prove that any subgroup H of (Z, +) is of the form <a> = aZ where a is a positive integer. (2)

   (c) If H = <a> and K = <b> are two subgroups of (Z, +), prove that H + K is a subgroup of (Z, +) and H + K = <d> where d = g.c.d.(a, b). (3)

P.T.O.

2. (a) Prove that order of a cyclic group is equal to order of its generator. (4)

(b) Prove that A_4 has no subgroup of order 6. (3)

(c) If N is a normal subgroup of G and $N \cap G' = \{e\}$, prove that $N \subseteq Z(G)$ where G' is commutator subgroup of G and Z(G) is the centre of G. (3)

SECTION II

3. (a) If G is the additive group of reals and N is the subgroup of G consisting of integers, prove that $\frac{G}{N}$ is isomorphic to the group H of all complex numbers of absolute value 1 under multiplication. (4)

(b) State Cayley's Theorem. Find the permutation group isomorphic to the Klein-4-group $G = \{e, a, b, ab\}$ where $a^2 = b^2 = e$ & $ab = ba$. (4)

(c) Let R' be the group of non-zero real numbers under multiplication, then prove that $(R', \cdot)$ is not isomorphic to $(R, +)$. (2)

4. (a) Let f be a homomorphism of G onto G' with kernel K and let N' be a normal subgroup of G'. Prove

that $\exists$ a normal subgroup N of G containing K such that

$$\frac{G}{N} \cong \frac{G'}{N'} \qquad (4)$$

(b) Let G be the group of all 2×2 matrices over reals of the type $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc \neq 0$ under matrix multiplication. Prove that the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to ad - bc$ is an onto homomorphism but not 1 – 1 from G to $(R - \{0\}, \cdot)$. (3)

(c) Using the result if G is a finite group and H is a subgroup of G, $H \neq G$ such that 0(G) does not divide (i(H))!, then H contains a nontrivial normal subgroup N of G", prove that if a group G of order 91 has a subgroup H of order 13, then H is normal in G. (3)

SECTION III

5. (a) Define an inner automorphism of a group G. Prove that the set 0(G) of all inner automorphisms of G is a normal subgroup of G. (4)

(b) Let σ & η be two conjugate permutations in S_n. Prove that they are similar.

Hence or otherwise prove that centre of S_n, $x \geq 3$ is trivial. (5)

6. (a) Let G be an infinite cyclic group. Determine Aut (G). (4)

(b) State and prove Cauchy's theorem for a finite abelian group. (5)

SECTION IV

7. (a) Define a p-group. Prove that a finite group G is a p-group iff $0(G) = p^n$ for some positive integer n. (5)

(b) Let G be a finite group in which $x^2 = e \ \forall \ x \in G$. Prove that $0(G) = 2^n$ for some integer n. Also prove that G is abelian. (4)

8. (a) Prove that a group of order 30 has normal subgroups of order 3, 5 and 15. (5)

(b) Determine all non-isomorphic abelian groups of order 8. (4)