This question paper contains 4 printed pages]

Roll No.

S. No. of Question Paper : **618**

Unique Paper Code : **235486** **E**

Name of the Paper : **Linear Algebra and Calculus**

Name of the Course : **B.A. (Hons.) Mathematics**

Semester : **IV**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *two* questions from each Section.

SECTION I

1. (*a*) In the vector space $\mathbf{R}^3$, let $W = \{(a, b, c) \to \mathbf{R}^3 : a - b - c = 0\}$. Prove that W is a subspace of $\mathbf{R}^3$. 5

(*b*) Define linearly independent subset of a vector space $\mathbf{R}^3$ over $\mathbf{R}$. Determine whether the following subset S of $\mathbf{R}^3$ is linearly independent or not.

$$S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}.$$ 5

(*c*) If V is a vector space over the field $\mathbf{F}$, then show that :

(*i*) $\alpha \cdot 0 = 0 \ \forall \ \alpha \in F$.

(*ii*) $0 \cdot v = 0 \ \forall \ v \in V$,

(*iii*) If $\alpha \cdot v = 0$, then $\alpha = 0$ or $v = 0$, $\forall \ \alpha \in F, v \in V$. 6

2. (*a*) Obtain a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ such that $T(1, 0, 0) = (1, 2)$, $T(0, 1, 0) = (2, 3)$ and $T(0, 0, 1) = (3, 4)$. 5

(*b*) Does the function $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(a, b) = (a + 1, b)\ \forall\ (a, b) \in \mathbf{R}^2$ a linear transformation ? Justify your answer. 5

(*c*) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be defined as $T(x, y, z) = (x + z, x + y, y - z)$. Verify Rank-Nullity Theorem for T. 6

3. (*a*) Let $u = (1, 3)$, $v = (2, 1) \in \mathbf{R}^2$. Find $w \in \mathbf{R}^2$ such that $\langle w, u\rangle = 3$, $\langle w, v\rangle = -1$, where $\langle , \rangle$ is the standard inner product on $\mathbf{R}^2$. 5

(*b*) State Cauchy-Schwarz Inequality and verify it for $u = (2, 1, 3)$, $v = (1, -1, 4) \in \mathbf{R}^3$. 6

(*c*) Find a vector of unit length which is orthogonal to the vector $(3, -2, 2)$ of $\mathbf{R}^3$ relative to standard inner product. 5

SECTION II

4. (*a*) Show that :

$$\lim_{x \to 0} \sin \frac{1}{x}$$

does not exist. 6

(*b*) Use $\varepsilon - \delta$ definition to prove that the following function is continuous at $x = 3$.

$$f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & , \quad x \neq 3 \\ 6 & , \quad x = 3 \end{cases}$$ 6

5. (*a*) Discuss the derivability of the function :

$$f(x) = \begin{cases} 2x - 3 & , \quad 0 \le x \le 2 \\ x^2 - 3 & , \quad 2 < x \le 4 \end{cases}$$

at $x = 2, 4$. 6

(*b*) If

$$f(x) = \begin{cases} x\dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Show that $f(x)$ is not derivable at $x = 0$. 6

6. (*a*) Verify Rolle's theorem for the function :

$$f(x) = x^2 - 6x + 8$$

in the interval [2, 4]. 6

(*b*) State and prove Lagrange's Mean value theorem. 6

SECTION III

7. For each of the limits,

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y) \text{ and } \lim_{y \to 0} \lim_{x \to 0} f(x, y) \text{ and } \lim_{(x,\, y) \to (0,\, 0)} f(x, y),$$

determine whether it exists or not, where :

$$f(x, y) = \frac{x^3 y}{2x^6 + y^2} \text{ whenever } (x, y) \neq (0, 0).$$ 9½

8. Let

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0) \end{cases}$$

Show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist but f is not differentiable at (0, 0). 9½

9. (*a*) Let f be a function defined as :

$$f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y} & , \quad x \neq y \\ 0 & , \quad x = y \end{cases}$$

Show that f possesses directional derivative in every direction at (0, 0). 4

(*b*) Locate all relative extrema and saddle points of :

$$f(x, y) = 4xy - x^4 - y^4.$$ 5½