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*Your Roll No.* ....................................

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**B.A./B.Sc. (Hons.)/III** **B**

MATHEMATICS—Paper XVI

(Analysis—V)

*Time : 2 Hours* *Maximum Marks : 38*

*(Write your Roll No. on the top immediately on receipt of this question paper.)*

Attempt any *two* parts from each question.

*All* questions carry equal marks.

1. (*a*) Prove that by an appropriate rearrangement of terms, a conditionally convergent series $\sum u_n$ can be made to converge to a preassigned number or can be made to diverge to $\infty$.

(*b*) (*i*) Examine the convergence of the series :

$$\sum_{n=2}^{\infty} \frac{(n^3+1)^{1/3}-n}{\log n}.$$

P.T.O.

(*ii*) Prove that :

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(m+n^2)(m+n^2-1)}=\frac{\pi^2}{12}.$$

(*c*) Show that for $|x|<1$

$$1+\frac{8x}{1-x}+\frac{16x^2}{1+x^2}+\frac{24x^3}{1-x^3}+\ldots\ldots\ldots\ldots$$

$$=1+\frac{8x}{(1-x)^2}+\frac{8x^2}{(1+x^2)^2}+\frac{8x^3}{(1-x^3)^2}+\ldots\ldots\ldots\ldots.$$

2. (*a*) Let a sequence $<f_n>$ of functions converge uniformly to a function $f$ on $I \sim \{c\}$. For all $n \in N$ let :

$$A_n = \lim_{x\to c} f_n(x)$$

Then :

(*i*) $<A_n>$ converges

(*ii*) $\lim_{x\to c} f(x) = \lim_{n\to\infty} A_n$.

(*b*) If a series $\sum f_n$ converges uniformly to a function $f$ in $[a, b]$ where each $f_n$ is continuous in $[a, b]$, then $f$ is continuous in $[a, b]$. What can you say about $f$ if the convergence is not uniform. Justify with an example.

(*c*) Examine the uniform convergence of the following :

(*i*) $\sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2} \quad x \in \mathbb{R}$

(*ii*) $< f_n(x) >$ where $f_n(x) = \frac{n^2 x}{1 + n^3 x^2}$ on $[0, 1]$.

3. (*a*) Expand in a series of sines and cosines of multiple angles of $x$, the periodic function $f$ with period $2\pi$ defined as :

$$f(x) = \begin{cases} -1 & \text{for} \quad -\pi < x < 0 \\ 1 & \text{for} \quad 0 \le x \le \pi \end{cases}$$

Calculate the sum of the series at $x = 0, \pi/2$.

(*b*) Examine whether the following series are Fourier series. Give the statement of the result that you use :

(*i*) $\frac{\sin x}{\sqrt{1}} + \frac{\cos 2x}{\sqrt{2}} + \frac{\sin 3x}{\sqrt{3}} + \frac{\cos 4x}{\sqrt{4}} + \ldots\ldots\ldots\ldots$

(*ii*) $\frac{\cos x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\cos 3x}{3^2} + \frac{\sin 4x}{4^2} + \ldots\ldots\ldots\ldots$.

(*c*) If $f$ is bounded and integrable in $[-\pi, \pi]$ and monotonic in $[-\delta, 0[$ and $]0, \delta]$ (not necessarily in the same sense) where $0 < \delta < \pi$ then :

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n = \frac{f(0-) + f(0+)}{\pi} \int_0^{\infty} \frac{\sin x}{x}\, dx$$

Hence deduce the value of $\int_0^{\infty} \frac{\sin x}{x}\, dx$

4. (*a*) Show that :

$$\sin^{-1} x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\,\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\,\frac{x^7}{7} + \ldots\ldots\ldots\ldots$$

$-1 < x \le 1$ and deduce that :

$$\frac{\pi}{2} = 1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1.3}{2.4}\,\frac{1}{5} + \frac{1.3.5}{2.4.6}\,\frac{1}{7} + \ldots\ldots\ldots\ldots$$

(*b*) (*i*) Show that both the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

and the corresponding series of derivatives

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

have the same radius of convergence.

(*ii*) Find the radius of convergence and the exact interval of convergence of the following power series :

$$\sum \frac{n+1}{(n+2)(n+3)} x^n .$$

(*c*) Define exponential function E($x$), cosine function C($x$) and sine function S($x$) as sums of power series. Prove that :

(*i*) $E(x) = e^x$ for all real $x$

(*ii*) $C(x + y) = C(x)\,C(y) - S(x)\,S(y)$.