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Your Roll No. ....................................

9664

# B.A./B.Sc. (Hons.)/III B

## MATHEMATICS—Unit XV

## (Analysis—IV)

*Time* : 2 *Hours* *Maximum Marks* : 38

*(Write your Roll No. on the top immediately on receipt of this question paper.)*

Attempt *Four* questions in all,

selecting *one* question from each Section.

### Section I

1. (*a*) If $f$ is bounded and integrable on $[a, b]$, then prove that $|f|$ is also integrable on $[a, b]$, and

$$\left|\int_a^b f\, dx\right| \leq \int_a^b |f|\, dx.$$ 3½

(*b*) Show that the function $f(x) = [x]$, where $[x]$ denotes the greatest integer not greater than $x$, is integrable in [0, 3], and

$$\int_0^3 [x]\,dx = 3.$$ 2½

(*c*) State the generalized form the first mean value theorem of integral calculus and verify it for the functions $f(x) = \sin x$ and $g(x) = e^x$ on the interval $[0, \pi]$. 3½

2. (*a*) Let

$$f(x) = \begin{cases} x & \text{if, } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$$

Show that $f$ is not Riemann integrable on [0, 1]. 3

(*b*) Prove that every continuous function is integrable. 3½ (*c*) Let $f(x) = x$ and $\alpha(x) = [x]$, $0 \le x \le 3$. Show that $f$ is Riemann-Stieltjes integrable with respect to $\alpha$ on [0, 3] and evaluate :

$$\int_0^3 f\, d\alpha. \qquad 3$$

## Section II

3. (*a*) Check the convergence of the improper integral :

$$\int_0^1 \frac{\log x}{\sqrt{x}}\, dx. \qquad 3$$

(*b*) Show that the improper integral :

$$\int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\right)\frac{e^{-x}}{x}\, dx$$

converges. 3½

(*c*) Show that :

$$\Gamma(2m) = \frac{1}{\sqrt{\pi}}\, 2^{2m-1}\, \Gamma(m)\, \Gamma\left(m + \frac{1}{2}\right). \qquad 3$$

4. (*a*) Check the convergence of improper integral :

$$\int_0^{\pi/3} \frac{\sqrt{x}}{\sin x}\, dx.$$ 3

(*b*) Show that :

$$\int_0^{\infty} \frac{\cos x}{\sqrt{x^2 + x}}\, dx$$

converges. 3½

(*c*) Evaluate :

$$\int_{-1}^{1} (1 + x)^{p-1} (1 - x)^{q-1}\, dx$$

in terms of Gamma function. 3

## Section III

5. (*a*) Evaluate the double integral :

$$\int_0^{2a} \int_{x^2/4a}^{3a - x} xy^2\, dy\, dx$$

by changing the order of integration. 3

(*b*) If $|x| < 1$, show that :

$$\int_0^{\pi/2} \log\left(a\cos^2\theta + b\sin^2\theta\right) d\theta = \pi \log\left[\frac{1}{2}\left(\sqrt{a} + \sqrt{b}\right)\right],$$

$a, b > 0$. 3

(*c*) Verify Green's theorem for

$$\oint_C \left(xy + y^2\right) dx + x^2\, dy$$

where C is the closed curve formed by $y = x$ and $y = x^2$. 3½

6. (*a*) By changing the order of integration, prove that :

$$\int_0^1 dx \int_x^{1/x} \frac{y^2 dy}{(x+y)^2 \sqrt{1+y^2}} = \frac{2\sqrt{2} - 1}{2}.$$ 3

(*b*) With the help of Green's formula, compute the difference between the line integrals :

$$I_1 = \int_{ACB} \left\{(x+y)^2 dx - (x-y)^2 dy\right\}$$

and $$I_2 = \int_{ADB} \left\{(x+y)^2 dx - (x-y)^2 dy\right\}$$

where ACB and ADB are respectively the straight line $y = x$ and the parabolic arc $y = x^2$, joining the points A(0, 0) and B(1, 1). 3½ (*c*) Evaluate the integral

$$I = \int_0^1 dx \int_0^x \sqrt{x^2 + y^2}\, dy$$

by passing on to the polar co-ordinates. 3

**Section IV**

7. (*a*) Find the area of that part of the surface of the cylinder $x^2 + y^2 = a^2$ which is cut out by the cylinder $x^2 + z^2 = a^2$. 4½

(*b*) State Stokes' theorem. Use it to find the line integral

$$\int x^2 y^3 dx + dy + z\, dz$$

where C is the circle $x^2 + y^2 = a^2$, $z = 0$. 5

8. (*a*) Evaluate the triple integral

$$\iiint_{\Omega} \frac{dx\,dy\,dz}{(x+y+z+1)^3}$$

where $\Omega$ is the interior of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. 4½

(*b*) Use Gauss theorem to evaluate

$$\iint_{S} x\,dy\,dz + y\,dz\,dx + z^2\,dx\,dy$$

where S denotes the closed surface bounded by the cone $x^2 + y^2 = z^2$ and the plane $z = 1$. 5