This question paper contains **4** printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **624**

Unique Paper Code : **235686** .E

Name of the Paper : **Linear Algebra and Calculus**

Name of the Course : **B.A. (Hons.) Mathematics-IV**

Semester : **VI**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *two* questions from each Section.

Section I

1. (*a*) Determine whether $W = \{(a, b, c) \in \mathbf{R}^3 : a + b + c = 0\} \subseteq \mathbf{R}^3$ is a subspace of $\mathbf{R}^3$ or not. 5

(*b*) Show that (3, –4, 6) belongs to subspace of $\mathbf{R}^3$ spanned by vectors (1, 2, –1), (2, 2, 1) and (1, –2, 3). 5

(*c*) Let $\mathbf{R}^2 = \{(a, b) : a, b \in \mathbf{R}\}$. Define addition and scalar multiplication as :

$$(a_1, a_2) + (b_1, b_2) = (0, a_2 + b_2)$$

$$\alpha(a_1, a_2) = (\alpha a_1, \alpha a_2), \alpha \in \mathbf{R}.$$

Show that $\mathbf{R}^2$ is not a vector space over $\mathbf{R}$. 6

2. (*a*) Let $f_1 = (1, 0), f_2 = (2, -1), f_3 = (4, 3)$ be three vectors in $\mathbf{R}^2$. Let $\{e_1, e_2, e_3\}$ be the standard basis for $\mathbf{R}^3$. If $T : \mathbf{R}^3 \to \mathbf{R}^2$ is a linear transformation such that $T(e_1) = (f_1)$, $T(e_2) = (f_2)$ and $T(e_3) = (f_3)$, find T(2, –3, 5). 5

(*b*) Does the function $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by

$T(x, y, z) = (x - y, x^2, 2z) \ \forall \ (x, y, z) \in \mathbf{R}^3$ a linear transformation ? Justify your answer. 5

(*c*) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear operator, the matrix A of which with resepct to standard basis is $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix}$. Verify Rank-Nullity Theorem for T. 6

3. (*a*) Show that the following is an inner product on $\mathbf{R}^2$, $\langle u, v \rangle = \alpha_1\beta_1 - 2\alpha_2\beta_1 - 2\alpha_1\beta_2 + 5\alpha_2\beta_2$, where $u = (\alpha_1, \alpha_2)$, $v = (\beta_1, \beta_2) \in \mathbf{R}^2$. 5

(*b*) Prove that :

$$\left\{\left(\frac{1}{3}, \frac{-2}{3}, \frac{-2}{3}\right), \left(\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)\right\}$$

form an orthonormal set in $\mathbf{R}^3$ w.r.t. standard inner product. Do they constitute a basis of $\mathbf{R}^3$? 6

(*c*) Find a vector of unit length which is orthogonal to the vector (3, –2, 2) of $\mathbf{R}^3$ relative to standard inner product. 5

Section II

4. (*a*) Use $\varepsilon - \delta$ definition to prove that : 6

$$\lim_{x \to 5} 3x = 15.$$

(*b*) Show that :

$$f(x) = \frac{1 - e^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}}$$

is discontinuous at $x = 0$. 6

5. (*a*) Discuss the derivability of the function :

$$f(x) = \begin{cases} x, & x < 1 \\ 2 - x, & 1 \le x \le 2. \\ -2 + 3x - x^2, & x > 2 \end{cases}$$

at $x = 1, 2$. 6

(*b*) Show that :

$$f(x) = |x - 1| + |x + 1|$$

is not derivable at $x = -1$ and 1. 6

6. (*a*) Show that there is no real number k for which the equation $x^3 - 3x + k = 0$ has two distinct roots in [0, 1]. 6

(*b*) State the intermediate value theorem for a continuous function. Let f be defined as follows :

$$f(x) = \begin{cases} x^2 + 1, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$

Show that f is not continuous at 0 and conclusion of intermediate value theorem does not hold. 6

Section III

7. Let 9½

$$f(x) = \begin{cases} xy\dfrac{(x^2 - y^2)}{(x^2 + y^2)}, & (x, y) \ne (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Show that :

$$f_{yx}(0, 0) \ne f_{xy}(0, 0).$$

8. Let $f(x, y) = |x| + |y|$ for all $(x, y) \in \mathbf{R}^2$. Show that f is continuous at (0, 0) but is not differentiate at (0, 0). 9½

9. (*a*) Let :

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Show that $f(x, y)$ has directional derivative in all direction at (0, 0). 4

(*b*) For the following function, locate all relative maxima, relative minima and saddle points if any :

$$f(x) = x^3 + y^3 - 3xy.$$ 5½