This question paper contains 7 printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **7851**

Unique Paper Code : **2351202** **F-II**

Name of the Paper : **Differential Equations-I (DC-1.4)**

Name of the Course : **Bachelor with Honours in Mathematics**

Semester : **II**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt *All* questions.

Use of scientific calculator is allowed.

Section I

1. Attempt any *three* of the following : 5+5+5

(*a*) Solve the initial value problem :

$$(2x + 3y + 1)dx + (4x + 6y + 1)dy = 0,\ y(-2) = 2.$$

(*b*) Solve the differential equation :

$$(x^2 - 3y^2)dx + 2xydy = 0.$$

(*c*) Solve the differential equation :

$$x\frac{dy}{dx} + y = -2x^6y^4.$$

(*d*) Check the exactness of the differential equation :

$$(5xy + 4y^2 + 1)dx + (x^2 + 2xy)dy = 0$$

hence solve it by finding an integrating factor.

2. Attempt any *one* of the following : 5

(*a*) The half-life of radioactive cobalt is 5.27 years. Suppose that a nuclear accident has left the level of cobalt radiation in a certain region at 100 times the acceptable for human habitation. How long will it be until the region is again habitable ? (Ignore the probable presence of other redioactive isotopes).

(*b*) A spherical tank of radius 4 ft is full of gasoline when a circular bottom hole with radius I in. is opened. How long will be required for all the gasoline to drain from the tank ?

Section II

3. Attempt any *two* of the following : 7½+7½

(*a*) The following model describes the levels of a drug in a patient taking a single cold pill :

$$\frac{dx}{dt} = -k_1 x, \qquad x(0) = x_0,$$

$$\frac{dy}{dt} = k_1 x - k_2 y, \qquad y(0) = 0$$

where k_1 and k_2 ($k_1 > 0$, $k_2 > 0$ and $k_1 \neq k_2$) describe the rates at which the drug moves between the two sequential compartments (GI-tract and blood stream). At time t, x and y are the levels of drug in GI-tract and blood stream respectively. Find solution expressions for x and y which satisfy this pair of differential equations.

(b) The following differential equation describes the level of pollution in the lake :

$$\frac{dC}{dt} = \frac{F}{V}(C_{in} - C)$$

where V is the volume of the lake, F is the flow (in and out), C is the concentration of pollution at time t and C_{in} is the concentration of pollution entering the lake. If $V = 28 \times 10^6$ m^3, $F = 4 \times 10^6$ m^3/month, find how long would it take for the lake with pollution concentration 10^7 parts/m^3 to drop below the safety threshold (4×10^6 parts/m^3) if

(i) Only fresh water enters the lake.

(ii) Water enters the lake has a pollution concentration of 3×10^6 parts/m^3

(c) A public bar opens at 6 pm and is rapidly filled with clients of whom the majority are smokers. The bar is equipped with ventilators which exchange the smoke-air mixture with fresh air. Cigarette smoke contains 4% carbon monoxide and a prolonged exposure to a concentration of more than 0.012% can be fatal. The bar has a floor area of 20 m by 15 m, and a height of 4 m. It is estimated that smoke enters the room at a constant rate of 0.006 m^3/min, and that the ventilators remove the mixture of smoke and air at 10 times the rate at which smoke is produced. The problem is to establish a wise time to leave the bar. That is, sometime before the concentration of carbon monoxide reaches the lethal limit.

Section III

4. Attempt any *two* of the following : 5+5

(*a*) Consider the differential equation :

$$x^2y'' - 3xy' + 4y = 0$$

Substitute $v = \ln x$ $(x > 0)$ in the above equation and solve.

(*b*) Show first that the two solutions :

$$y_1(x) = e^x \cos x,\ y_2(x) = e^x \sin x$$

of the differential equation :

$$y'' - 2y' + 2y = 0$$

are linearly independent on the open interval I. Then find a particular solution of above differential equation with the initial conditions $y(0) = 1$, $y'(0) = 5$.

(*c*) Find a general solution of the differential equation :

$$(D^2 + 6D + 13)^2\, y = 0,$$

where

$$D = \frac{d}{dt}.$$

5. Attempt any *two* of the following : 5+5

(*a*) Find the general form of a particular solution y_p of the differential equation :

$$y^{(3)} + y' = x \sin x + x^2 e^{2x}$$

using the method of undetermined coefficients.

(*b*) Use the method of variation of parameters to find a particular solution of the differential equation :

$$y'' - 4y = xe^x.$$

(*c*) Use the method of undetermined coefficients to find the particular solution of the differential equation :

$$3y'' + y' - 2y = 2 \cos x.$$

Section IV

6. Attempt any *two* of the following : 10+10

(*a*) A simple model for a battle between two armies red and blue, where both the armies used aimed fire, is given by the coupled differential equations :

$$\frac{dR}{dt} = -a_1 B, \frac{dB}{dt} = -a_2 R,$$

where R and B are the number of soldiers in red and blue armies respectively, a_1 and a_2 are positive constants.

If both the armies have equal attrition coefficients i.e. $a_1 = a_2$ and there are 10000 soldiers in red army and 8000 in blue army, determine who wins, if :

(*i*) There is one battle between the two armies.

(*ii*) There are two battles, first battle with half the red army against the entire blue army and second with the other half of the red army against the blue army survivors of the first battle.

(*b*) Consider the Lotka-Volterra model describing the simple predator-prey model :

$$\frac{dX}{dt} = b_1 X - c_1 XY, \frac{dY}{dt} = c_2 XY - a_2 Y,$$

where b_1, c_1, c_2 and a_2 are positive constants, X denotes the prey population and Y denotes the predator population at time t.

(*i*) Find the equilibrium solutions of the above model.

(*ii*) Find directions of trajectories in the phase plane.

(*c*) A model for the spread of a disease, where once susceptible infected, confers life-long immunity, is given by the coupled differential equations :

$$\frac{dS}{dt} = -\beta SI, \frac{dI}{dt} = \beta SI - \alpha I$$

where α and β are positive constants, S(t) denotes the number of susceptibles and I(t) denotes the number of infectives at time t :

(i) Use the chain rule to find a relation between S and I, given the initial number of susceptible and infectives are s_0 and i_0, respectively.

(ii) Find and sketch directions of trajectories in the phase plane.