This question paper contains 4+1 printed pages]

Roll No. | | | | | | | | | | |

S. No. of Question Paper : **8132**

Unique Paper Code : **235603** **D**

Name of the Paper : **Algebra - V**

Name of the Course : **B.Sc. (H) Mathematics**

Semester : **VI**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *two* parts from each question.

All questions are compulsory.

1. (*a*) How many subgroups of order 4 does $Z_4 \oplus Z_2$ have ? Justify.

(*b*) Prove that if a group G is the internal direct product of a finite number of subgroups $H_1, H_2, \ldots\ldots, H_n$, then G is isomorphic to the external direct product of $H_1, H_2, \ldots\ldots, H_n$.

(*c*) The set {1, 9, 16, 22, 29, 53, 74, 79, 81} is a group under multiplication modulo 91. Determine the isomorphism class of this group. Also, express G as an internal direct product of cyclic groups. [6.5,6.5,6.5]

2. (*a*) Let G be a group, let H be a subgroup of G and let G act by left multiplication on the set A of left cosets of H in G. Let φ be the associated permutation representation afforded by this action. Then show that :

(*i*) G acts transitively on A

(*ii*) the stabilizer in G of $eH \in A$ is the subgroup H

(*iii*) the kernel of φ is the largest normal subgroup of G contained in $H \ni \ker \varphi = \bigcap_{x \in G} xHx^{-1}$.

(*b*) (*i*) If G is a group of order p^n for some prime number p and $n \geq 1$, then show that G has a non-trivial centre.

(*ii*) Prove that the kernel of an action of the group G on the set A is the same as the kernel of the corresponding permutation representation $G \to S_A$.

(*c*) Define a simple group. Show that A_5 is the only non-trivial proper normal subgroup of S_5. [7,7,7]

3. (*a*) Let $|G| = pq$, where p, q are distinct primes, $p < q$, $p \nmid (q - 1)$. Show that G is cyclic.

(*b*) (*i*) Suppose that G is a finite simple group and contains subgroups H and K such that $|G : H|$ and $|G : K|$ are prime. Show that $|H| = |K|$.

(*ii*) Prove that a group of order 210 cannot be simple.

(*c*) Let $|G| = 30$. Show that both Sylow 3-subgroup and Sylow 5-subgroup are normal in G. [7,7,7]

4. (*a*) Find an orthogonal matrix whose first row is :

$$\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right).$$

(*b*) Let T be a normal operator on an inner product space V over the field F. Then the following statements are equivalent :

(*i*) $\|Tx\| = \|T^*x\| \quad \forall\ x \in V$

(*ii*) $T - cI$ is normal $\forall\ c \in F$

(*iii*) If x is an eigen vector of T, then x is also an eigen vector of T^*.

(*iv*) If λ_1 and λ_2 are distinct eigen values of T with corresponding eigen vectors x_1 and x_2, then x_1 and x_2 are orthogonal.

(*c*) Let T be a linear operator on the finite dimensional complex inner product space V. Then V has an orthonormal basis of eigen vectors of T with corresponding eigen values of absolute value 1 if and only if T is unitary. [6,6,6]

5. (*a*) Find new coordinates x', y' so that the quadratic form :

$$2x^2 + 2xy + 2y^2 = 36$$

can be written as :

$$\lambda_1(x')^2 + \lambda_2(y')^2 = 36.$$

(*b*) Let T be a linear operator on R^3 which is represented in the standard ordered basis by the matrix :

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}.$$

Show that $\exists$ a diagonalizable operator D on R^3 and a nilpotent operator N on R^3 such that T = D + N and DN = ND. Find the matrices of D and N in the standard basis.

(*c*) If T is a linear operator on a finite dimensional vector space V, then prove that T has a cyclic vector if and only if there is some ordered basis for V in which T is represented by the companion matrix of the minimal polynomial for T. [5,5,5]

6. (*a*) Prove that if T^2 has a cyclic vector, then T has a cyclic vector. Is the converse true ? Justify.

(*b*) Let T be a linear operator on R^3 which is represented in the standard ordered basis by the matrix :

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

Find its Rational form and hence find an invertible 3×3 real matrix P such that $P^{-1}AP$ is in rational form.

(*c*) How many possible Jordan forms are there for a 3×3 complex matrix A given by :

$$A = \begin{bmatrix} 4 & 0 & 0 \\ a & 4 & 0 \\ a & b & -2 \end{bmatrix}$$

Under what condition A is similar to a diagonal matrix. [6,6,6]