This question paper contains 7 printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **8131**

Unique Paper Code : **235601** **D**

Name of the Paper : **Analysis–V (6.1)**

Name of the Course : **B.Sc. (Hons.) Mathematics**

Semester : **VI**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

All questions are compulsory.

Attempt any *two* parts from each question.

1. (*a*) Define region and show that the region S is polygonally connected.

(*b*) (*i*) Let Arg(w) denotes the values of the argument between $-\pi$ and π (inclusive).

Show that :

$$\operatorname{Arg}\left(\frac{z-1}{z+1}\right) = \pi/2 \quad \text{if} \quad \operatorname{Im} z > 0$$
$$= -\pi/2 \quad \text{if} \quad \operatorname{Im} z < 0$$

P.T.O.

(*ii*) Show that the function $f(z) = \bar{z}$, where $\bar{z}$ is the complex conjugate of z, is nowhere differentiable.

(*c*) (*i*) Show that there is no power series :

$$f(z) = \sum_0^\infty a_n z^n$$

such that :

$f(z) = 1$ for $z = 1/2, 1/3, 1/4, \ldots$ and $f'(0) > 0$.

(*ii*) If f is analytic in a region and if $|f|$ is constant there, then show that f is constant. 6,4+2,3+3

2. (*a*) (*i*) Suppose $T \subset \mathbb{C}$ show that the corresponding set S on the Riemann Sphere Σ :

(1) is a circle if T is a circle.

(2) a circle minus (0, 0, 1) if T is a line.

(*ii*) Show that the function $f(z) = |z|^2$ is differential only at $z = 0$ and $f'(0) = 0$.

(*b*) (*i*) Find all the solutions of $e^z = 1$.

(*ii*) Show that the functions :

$$f(z) = f(x+iy) = xy(x+iy)/(x^2+y^2) \quad \text{if} \quad x+iy \neq 0$$
$$= 0 \quad \text{if} \quad x+iy = 0$$

satisfies the Cauchy-Riemann Equations at $z = 0$ but not differentiable there.

(*c*) (*i*) Suppose that g is the inverse of f at z_0 and that g is continuous there. If f is differentiable at $g(z_0)$ and if $f'(g(z_0)) \neq 0$, then g is differentiable at z_0 and $g'(z_0) = 1/f'(g(z_0))$.

(*ii*) Find the radius of convergence of the series :

$$\sum_{n=0}^{\infty} \left(\frac{n!z^n}{n^2} \right).$$

4+2,2+4,4+2

3. (*a*) (*i*) Let C be a smooth curve given by $z(t)$, $a \le t \le b$, and suppose f is continuous at all the points $z(t)$. Then prove that :

$$\int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz,$$

where $-C$ is the opposite curve of the curve C given by :

$$-C : w(t) = z(a+b-t).$$

(ii) Evaluate :

(1) $\int_{C_1} (z-z^2)\,dz,$ where $c_1 : z(t) = (1-t) + it; 0 \le t \le 1$

(2) $\int_{C_2} \frac{1}{(z-2)}\,dz,$ where $c_2 : z(t) = 2 + e^{it}\ ;\ 0 \le t \le \pi.$

(b) (i) Show that :

$$\int_C z^k\,dz = 0$$

for any integer $k \neq -1$ and $C : z(t) = Re^{it}; 0 \le t \le 2\pi$

(1) By showing that z^k is the derivative of a function analytic throughout C.

(2) Directly using the parameterization of C.

(ii) Find the power series expansion of $f(z) = z^2$ around $z = 2$.

(c) (i) State and prove the Liouville's Theorem.

(ii) Suppose f entire and $|f'(z)| \le |z|, \forall z.$

Show that :

$$f(z) = a + bz^2 \text{ with } |b| \le \frac{1}{2}.$$

2+4,4+2,4+2

4. (*a*) Suppose that C is a (smooth) curve of length L and f is continuous on C and :

$$|f(z)| \leq \mathbf{M} \ \forall z \in \mathbb{C}.$$

Then prove that :

$$\left| \int_C f(z) dz \right| \leq \mathbf{ML}.$$

(*b*) Show that if f is continuous real valued function and $|f(z)| \leq 1$, then show that :

$$\left| \int_{|z|=1} f(z) dz \right| \leq 4.$$

(*c*) If f is entire and

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & z \neq a \\ f'(a) & z = a \end{cases}$$

then prove that :

$$\int_\Gamma g(z)\, dz = 0,$$

where Γ is the boundary of a rectangle R. 6,6,6

5. (*a*) If f is analytic in $D(\alpha; r)$, then show that there exist constants c_k such that :

$$f(z) = \sum_{k=1}^{\infty} c_k (z-\alpha)^k, \ \forall z \in D(\alpha; r).$$

(*b*) Find the Laurent Expansions for :

(*i*) $\dfrac{1}{z^2(z-1)}$ about $z = 1$

(*ii*) $\dfrac{1}{z^4+z^2}$ about $z = 0$

(*iii*) $\dfrac{1}{z^2-4}$ about $z = 2$.

(*c*) Prove that the bilinear transformation $f(z) = 1/z$ maps a circle to a circle or a line.

7.5,2.5×3,7.5

6. (*a*) (*i*) Find the fundamental period of :

$$\cos\left(\frac{2\pi nx}{k}\right) \text{ and } \sin(nx).$$

(*ii*) Are the following functions $f(x)$ which are assumed to be periodic, of period 2π, even, odd or neither even nor odd ?

(1) $f(x) = \cos^2(x)$ if $-\pi < x < 0$,

$= \sin^2(x)$ if $0 < x < \pi$.

(2) $f(x) = x$ if $0 < x < \pi$,

$= \pi - x$ if $\pi < x < 2\pi$.

(*b*) Find the Fourier series of :

$$f(x) = |x| \text{ on } [-2, 2] \text{ and } f(x+4) = f(x) \ \forall x.$$

(*c*) Find the Fourier cosine series as well as the Fourier sine series of the function $f(x)$ given by :

$$f(x) = x^2, \ 0 < x < L.$$

Deduce $1 + 1/2^2 + 1/3^2 + ... = \pi^2/6.$ 2+2×2,6,6