This question paper contains 4 printed pages]

Your Roll No. ....

6734

B.A./B.Sc. (Hons.)/I **D**

MATHEMATICS—Unit IV

(Analysis—II)

(Admissions of 2008 and before)

Time : 2 *Hours* *Maximum Marks* : 50

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *two* parts from each question.

1. (*a*) If a function f is continuous in $[a, b]$ and $f(a)\,f(b) < 0$, then prove that there exists $x_0 \in\ ]a, b[$ such that $f(x_0) = 0$. 6

(*b*) Examine the continuity of the function f defined by :

$$f(x) = \lim_{n \to \infty}\left(\frac{e^x - x^n \sin x}{1 + x^n}\right) \quad \left(0 \le x \le \frac{\pi}{2}\right)$$

at $x = 1$. Explain why the function f does not vanish anywhere in $\left[0, \frac{\pi}{2}\right]$, although $f(0)\ f\left(\frac{\pi}{2}\right) < 0$. 6

(*c*) Show that every uniformly continuous function is continuous. Give an example of a continuous function which is not uniformly continuous. 6

2. (*a*) State and prove Cauchy's Mean Value Theorem. Using it show that :

$$\sin\alpha - \sin\beta = \cot\theta\ (\cos\beta - \cos\alpha)$$

where $< \alpha < \theta < \beta < \pi/2$. 7

(*b*) Prove that if a function is derivable at a point, then it is continuous at that point. By considering the function f defined by :

$$f(x) = \begin{cases} x\left[1 + \frac{1}{3}\sin\left(\log x^2\right)\right] & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

show that converse is not true. 7

(*c*) If $f(0) = 0$ and $f''(x)$ exists on $[0, \infty[$, show that :

$$f'(x) - \frac{f(x)}{x} = \frac{1}{2}xf''(c), \quad 0 < c < x$$

Also deduce that if $f''(x)$ is positive for positive values of x, then $f(x)/x$ is strictly increasing in $]0, \infty[$.

3. (*a*) Obtain the Maclaurin's series expansion of $\log(1 + x)$. 6

(*b*) Find the value of a and b in order that :

$$\lim_{x \to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3}$$

is equal to 1. 6

(*c*) Show that :

$$\frac{x}{1 + x} < \log(1 + x) < x$$

For $x > -1$ and $x \neq 0$. 6

4. (*a*) Evaluate any *two* of the following :

(*i*) $\int \frac{dx}{(x^2 + 2)\sqrt{x^2 - 3}}$

(*ii*) $\int \frac{dx}{\sqrt{1 + \sqrt{x}}}$

(*iii*) $\int \frac{\sin^6 x \, dx}{\cos^4 x}$. 6

(*b*) Show that for $m, n \in \mathbb{N}$,

$$\int_0^{\pi/2} \cos^m x \cos nx \, dx = \frac{m}{n} \int_0^{\pi/2} \cos^{m-1} x \cos(n-1)x \, dx.$$

Also deduce that :

$$\int_0^{\pi/2} \cos^6 x \cos 6x \, dx = \frac{\pi}{128}.$$ 6

(*c*) Find the whole area between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes. 6