This question paper contains 4+2 printed pages]

Your Roll No. ....

6735

B.A./B.Sc. (Hons.)/II **D**

MATHEMATICS—Unit V

(Algebra—II)

(Admissions of 2008 and before)

Time : 2 Hours *Maximum Marks :* 38

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *one* question from each Section.

Section I

(*a*) G = Set of rational numbers.

Is G a group under *, where :

$$a * b = \frac{ab}{3} \ \forall \ a, b \in G$$ 3

(*b*) A finite semi group in which cancellation laws hold is a group. Prove it. 4

(c) Let = $\left\{\begin{bmatrix} a & a \\ b & b \end{bmatrix} : a, b \in \mathbb{R}, a + b \neq 0\right\}$

Show that M is a semi group under matrix multiplication and has a right identity and a left inverse for each element. 3

2. (a) Prove that a non-empty subset H of a group G is a subgroup of G iff :

$$a, b \in H \Rightarrow ab^{-1} \in H$$ 4

(b) Find the order of each element of the quarternion group :

$$Q^* = \{\pm 1, \pm i, \pm j, \pm k\}.$$

Is this group cyclic ? 3

(c) Find all the generators of a cyclic group G of order 20. 3

Section II

3. (*a*) Prove that if H and K be two subgroups of a finite group G then :

$$o(HK) = \frac{o(H)\, o(K)}{o(H \cap K)}$$ 4

(*b*) Let H and K be two normal subgroups of a group G s.t. $H \cap K = \{e\}$ then show that :

$$hk = kh \;\forall\; h \in H \text{ and } k \in K.$$ 3

(*c*) Let Z(G) be the centre of a group G, prove that Z(G) is a normal subgroup of G. 3

4. (*a*) Write all the elements of the quotient group **Z**/5**Z** of **Z**. Is it cyclic ? Explain. 3

(*b*) Show that the mapping of $f : \mathbf{C} - \mathbf{C}$ of complex numbers defined by :

$$f(Z) = \overline{Z} \quad \forall\, Z \in \mathbf{C},$$

is an automorphism of **C**. 3

(*c*) S_n = Permutation group of n elements

A_n = Set of all even permutations of S_n

Prove that :

$S_n / A_n \cong \{1, -1\}$ = group of square roots of unity.

Section III

5. (*a*) Show that $\forall\, n \geq 3$, the subgroup generated by 3-cycles of S_n is A_n. 3

(*b*) Prove that for any group G,

$G / Z(G) \cong I(G)$ = Set of all inner automorphisms of G. 3

(*c*) Determine Aut(G) if G is an infinite cyclic group. 3

6. (*a*) Let G be a group of order p^n, p-prime then show that :

$$o(Z(G)) > 1.$$ 4

(*b*) Prove that number of conjugate classes in S_n is $p(n)$, the number of partitions of n. 5

Section IV

7. (*a*) State Sylow's three theorems and verify them on S_3. 5

(*b*) Find all the different groups of order 4. Hence or otherwise prove that every group of order 4 is abelian. 4

8. (*a*) Let G be a group and it is IDP of $H_1, H_2, , H_n$, prove that G is isomorphic to EDP of $H_1, H_2, , H_n$. 5

(*b*) Find all the non-isomorphic abelian groups of order :

(*i*) 8

(*ii*) 20. 4