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Your Roll No. ....

6742

B.A./B.Sc. (Hons.)/III **D**

MATHEMATICS—Unit 12

(Algebra—III)

Time : 2 *Hours* *Maximum Marks* : 38

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *two* parts from each Section.

Section I

1. (*a*) For any prime integer p, show that the ring z_p, of integers modulo p, is a field and conversely. 4½

(*b*) Let R be the ring of all the real valued, continuous function on a closed unit interval :

$$\text{Let } M = \left\{ f(x) \in R \middle| f\left(\frac{1}{5}\right) = 0 \right\}.$$

Show that M is a maximal ideal of R. 4½

P.T.O.

(*c*) If x is a non-zero element of an integer domain R such that $mx = 0$ for some positive integer m, prove that characteristics of R is finite. Also, show that characteristics of R divides m. 4½

Section II

2. (*a*) If A and B are ideal of a ring R, prove that : 5

$$\frac{A + B}{A} \cong \frac{B}{A \cap B}.$$

(*b*) (*i*) Let D be an integral domain. If $a, b \in D$ are such that $a^n = b^n$ and $a^m = b^m$, where m, n are relatively prime positive integers, then prove that $a = b$.

(*ii*) Let R be an integral domain and let F be its quotient field. Let $0 \neq a \in R$. Show that the mapping $\Phi : R \rightarrow F$ defined by $\Phi(x) = [xa, a] \ \forall \ x \in R$ is an isomorphism of R into F. 5

(c) (i) Let R be a ring with unity 1 and let Φ be a homomorphism from R into an integral domain R′ such that kernel (Φ) $\neq$ R. Show that $\Phi(1)$ is unity of R′.

(ii) Let P be a prime ideal of a Boolean ring R such that P $\neq$ R, show that P is a maximal ideal of R. 5

Section III

3. (a) (i) Let R be a Euclidean domain. Let a and b non-zero elements of R. Prove that $d(ab) = d(a)$, if and only if b is invertible.

(ii) State 'Einstein Criterion' for the irreducibility of a polynomial with integral coefficients over the field Q of rational numbers. Use it to show that the polynomial $1 + x + x^2 + x^3 + x^4$ is irreducible over Q. 5

(*b*) Let R be a UFD (Unique Factorizational Domain) and $f(x) \in R[x]$ be an irreducible element of $R[x]$. Prove that either f is an irreducible element of R or f is an irreducible primitive polynomial of $R[x]$. 5

(*c*) In $Z[i]$ find the g.c.d. of $11 + 7i$, $3 + 7i$. 5

Section IV

4. (*a*) Prove that an element $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F. 4½

(*b*) Prove that : 4½

$$Q\left(\sqrt{2}, 3\right) = Q\left(\sqrt{2} + \sqrt{3}\right).$$

(*c*) Show that it is impossible to duplicate the cube by using ruler and compass. 4½