This question paper contains 4+2 printed pages]

Your Roll No. ....

6743

B.A./B.Sc. (Hons.)/III **D**

MATHEMATICS—Unit XIII

(Algebra—IV)

Time : 2 *Hours* *Maximum Marks* : 38

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt *two* questions from each Section.

Section I

1. (*a*) Define the linear span L(S) of a subset S of a vector space V(F). 2.5

Let $V = \mathbf{R}^3(\mathbf{R})$, $S = \{\alpha = (1, 1, 0),$

$\beta = (0, -1, 1), \gamma = (1, 0, 1)\}$

Prove that :

$(a, b, c) \in L(S)$

if and only if $a = b + c$.

(*b*) Prove that :

$$S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1), (0, 1, 0)\}$$

is a spanning set of $\mathbf{R}^3$ but is not a basis of $\mathbf{R}^3$. 2

2. Let A and B be two subspaces of a finite dimensional vector space V(F). 4.5

Prove that :

$$\dim(A + B) = \dim A + \dim B - \dim (A \cap B).$$

Hence find the dimension of the subspace A + B of $\mathbf{R}^3(\mathbf{R})$,

where $A = \{(x, y, o) : x, y \in \mathbf{R}\}$ and

$$B = \{(o, y, z) : y, z \in \mathbf{R}\}$$

3. Let W be a subspace of a finite dimensional vector space *V(F)*, *prove that* there exists a subspace W' of V such that V is direct sum of W and W'. 4.5

Section II

4. (*a*) Let a linear transformation $T : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ be defined by :

$$T(x, y, z) = (x + y, y + z, z + x)$$

Is T invertible ? If yes, find $T^{-1}(a, b, c)$. 3

(*b*) Give an example of a linear operator T on $\mathbf{R}^2$ such that : 2

$$T^2 = 0 \text{ but } T \neq 0.$$

5. Define a non-singular linear transformation. :

$$\text{Let } T : V(F) \rightarrow W(F)$$

be a linear transformation where dim V = dim W. Prove that T is non-singular if and only if T maps a basis of V onto a basis W. 5

6. Let T be a linear operator on $\mathbf{R}^2$ defined by :

$$T(x, y) = (x, x + y).$$

Let $\beta = \{e_1 = (1, 0), e_2 = (0, 1)\}$ and

$$\beta' = \{\alpha_1 = (1, 2), \alpha_2 = (1, 1)\}$$

be two ordered bases for $\mathbf{R}^2$. Find a matrix P such that : 5

$$[T]_{\beta'} = P^{-1}[T]_{\beta}P.$$

Section III

7. State and prove Cauchy-Schwarz Inequality and find necessary and sufficient conditions for this inequality to be an equality. 4.5

8. Define annihilator A(W) of a subspace W of a vector space V(F). If V(F) is finite dimensional, prove that : 4.5

$$\dim A(W) = \dim V - \dim W.$$

9. (a) If V is a finite dimensional vector space over F and $v_1 \neq v_2$ are in V, show that there exists some $f \in \tilde{V}$ such that : 2

$$f(v_1) \neq f(v_2).$$

(b) Let V be the space of real functions satisfying :

$$\frac{d^2y}{dx^2} + 9y = 0.$$

In V, inner product is defined by $\langle y, z \rangle = \int_0^{\pi} yz \, dx$.

Prove that dimension of V is 2 and obtain an orthonormal basis of V. 2.5

Section IV

10. (a) Prove that the characteristic vectors corresponding to distinct characteristic values of T are linearly independent. 3

(*b*) Let T be the linear operator on $\mathbf{C}^2$ defined by :

$$T(e_1) = (0, 1) \text{ and } T(e_2) = (-1, 0)$$

Find the minimal polynomial for T. 2

11. Find the characteristic values and basis of the corresponding characteristic spaces of the matrix A, whose column vectors are (2, 0, 0), (1, 1, 2) and (0, –1, 4). Is A similar to a diagonal matrix ? Give reasons. 5

12. (*a*) Let E be a projection on V and let T be a linear operator on V. Prove that the range of E is invariant under T if and only if ETE = TE. 3

(*b*) Prove that both the range and null space of E are invariant under T if and only if ET = TE. 2